

Continuous Spectrum of Automorphism Groups and the Infraparticle Problem

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Abstract

This paper presents a general framework for a refined spectral analysis of a group of isometries acting on a Banach space, which extends the spectral theory of Arveson. The concept of continuous Arveson spectrum is introduced and the corresponding spectral subspace is defined. The absolutely continuous and singular-continuous parts of this spectrum are specified. Conditions are given, in terms of the transposed action of the group of isometries, which guarantee that the pure-point and continuous subspaces span the entire Banach space. In the case of a unitarily implemented group of automorphisms, acting on a C^* -algebra, relations between the continuous spectrum of the automorphisms and the spectrum of the implementing group of unitaries are found. The group of spacetime translation automorphisms in quantum field theory is analyzed in detail. In particular, it is shown that the structure of its continuous spectrum is relevant to the problem of existence of (infra-)particles in a given theory.

1 Introduction

In the familiar case of a strongly continuous group of unitaries $U(t) = e^{iHt}$, acting on a Hilbert space \mathcal{H} , spectral theory is well understood. In particular, \mathcal{H} can be decomposed into the pure-point, absolutely continuous and singular-continuous subspaces, which reflects the decomposition of the spectral measure of H into measure classes. On the other hand, for a strongly continuous group of isometries $\alpha_t = e^{iDt}$ acting on a Banach space \mathfrak{A} , spectral theory is much less developed. While the Arveson spectral theory provides subspaces associated with closed subsets of the spectrum of D [Ar82, Ev76, Lo77], there does not seem to exist any general definition of the continuous spectral subspace, not to speak of its absolutely continuous or singular continuous parts. It is the main goal of the present paper to introduce such

notions and demonstrate their relevance to the problem of particle interpretation in quantum field theory (QFT).

In the Hilbert space setting these detailed spectral concepts provided a natural framework for the formulation and resolution of the problem of asymptotic completeness in quantum mechanics [En78, SiSo87, Gr90, De93]. The absence of such structures on the side of Banach spaces impedes the study of particle aspects in quantum field theory, where the time evolution is governed by a group of automorphisms α acting on a C^* -algebra of observables \mathfrak{A} . Asymptotic completeness is an open problem in all known models of interacting quantum fields, except for a recently constructed class of two-dimensional theories with factorizing S -matrices [Le08]. Since pairs of charged particles may be produced in collisions of neutral particles, it is not even *a priori* clear what particle types a given theory describes. Finally, the possible presence of charges with weak localization properties, like the electric charge in quantum electrodynamics, forces one to depart from the conventional Wigner concept of a particle as a state in some irreducible representation space of the Poincaré group [Wi39]. In spite of decades of research [Sch63, FMS79, Bu86], this *infraparticle problem* is still a largely open issue.

One approach to the problems listed above is to study simplified models which capture some relevant features of quantum field theories. Over the last decade there has been significant progress along these lines [DG00, FGS04, Sp, Pi05, CFP07, He07, Re09]. A complementary approach, pursued in algebraic quantum field theory, aims at a development of a model-independent concept of a particle which is sufficiently general to encompass all the particle-like structures appearing in quantum field theory [Bu94]. Substantial steps in this direction were made by Buchholz, Porrmann and Stein [BPS91]. In order to clarify the relation between the particle aspects of quantum field theory and the (Arveson) spectrum of the group of automorphisms $\mathbb{R}^{s+1} \ni (t, \vec{x}) \rightarrow \alpha_{(t, \vec{x})}$, which describes spacetime translations of observables, we recall the main steps of this analysis: To extract the particle content of a physical state $\omega \in \mathfrak{A}^*$, one has to compensate for dispersive effects. To this end, one paves the whole space with observables and sums up the results. This amounts to studying the time evolution of the integrals

$$\sigma_\omega^{(t)}(A) := \int d^s x \omega(\alpha_{(t, \vec{x})}(A)), \quad (1.1)$$

where $A \in \mathfrak{A}$ is a suitable observable. The limit points of $\sigma_\omega^{(t)}$ as $t \rightarrow \infty$, called the asymptotic functionals $\sigma_\omega^{(+)}$, carry information about all the particle types appearing in the theory. In fact, for theories of Wigner particles it was shown in [AH67] that each asymptotic functional can be represented as a mixture of plane wave configurations of all the particle types contributing to the state ω

$$\sigma_\omega^{(+)}(A) = \sum_\lambda \int d^s p \rho_\lambda(\vec{p}) \langle \vec{p}, \lambda | A | \vec{p}, \lambda \rangle. \quad (1.2)$$

This mixture is labeled by $\lambda = [m, s, q]$ (i.e. mass, spin and charge), possibly including pairs of charged particles. The functions ρ_λ stand for the asymptotic

densities of the respective particle types. In the general case, including both Wigner particles and infraparticles, a similar expression was derived by Porrmann [Po04.2]

$$\sigma_{\omega}^{(+)}(A) = \int d\mu(\lambda) \sigma_{\lambda}^{(+)}(A). \quad (1.3)$$

Here the analogues of the plane wave configurations are the so-called pure particle weights $\sigma_{\lambda}^{(+)}$ labeled by $\lambda = [p, \gamma]$, where p is the four-momentum of the particle and γ carries information about the other quantum numbers, like spin or charge. Each pure particle weight gives rise to an irreducible representation of \mathfrak{A} . In contrast to the previous case, the representations corresponding to different four-momenta p may be inequivalent. Thus the infraparticle situation, e.g. the electron whose velocity gives a superselection rule [FMS79, Bu82, Bu86], can be treated in this framework.

A necessary prerequisite for this approach is the existence of non-zero asymptotic functionals. It is well known from the study of generalized free fields that this property does not follow from the general postulates of quantum field theory. However, it has been established for theories of Wigner particles [AH67] and in a non-interacting model of an infraparticle introduced by Schroer [Sch63, Joh91]. Moreover, in [Dy08.2] we supplied a model-independent argument, ensuring the existence of non-zero asymptotic functionals in a certain class of theories containing a stress-energy tensor¹. In view of this recent progress, one can hope for a general classification of quantum field theories w.r.t. their particle structure. It is the main goal of the present work to develop a natural language for such a classification.

As a first step in this direction, we infer from formula (1.2) that the asymptotic functional is non-trivial, only if its domain contains sufficiently many observables, whose energy-momentum transfer includes zero. We recall that the energy-momentum transfer of an observable A coincides with the Arveson spectrum of the group of automorphisms α restricted to the subspace spanned by the orbit of A (cf. definition (2.3)). Therefore, essential information about the particle content of a given theory should be encoded in the properties of the Arveson spectrum of α in a neighborhood of zero. As it is evident from definition (1.1) that the joint eigenvectors of the generators of α do not belong to the domains of the asymptotic functionals, the relevant part of the spectrum is the continuous one.

However, the existing spectral theory is not yet sufficiently developed to test such fine features of the Arveson spectrum. It provides a functional calculus and spectral mapping theorems for some classes of functions as well as a definition and properties of spectral subspaces associated with closed subsets of the spectrum. (See [Ar82] for a review). This allows e.g. for a study of the pure-point spectrum which has attracted much attention from various perspectives [Ba78, Jo82, AB97, Hu99]. To our knowledge, there is no systematic analysis of the continuous Arveson spectrum in the literature. Nevertheless, there exist some interesting results pertaining to decay properties of the functions $\mathbb{R}^d \ni x \rightarrow \omega(\alpha_x(A))$ and regularity properties of their Fourier transforms $\mathbb{R}^d \ni p \rightarrow \omega(\tilde{A}(p))$. We mention the analysis of Jorgensen

¹The relevance of the stress-energy tensor to particle aspects was first pointed out in [Bu94].

[Jo92], inspired by the Stone formula, and a result of Buchholz [Bu90], concerning space translations in QFT, which will be used in Section 5 of the present paper. The Fourier transforms $\omega(\tilde{A}(\cdot))$ appear also as a tool in the literature related to the Rieffel project of extending the notions of proper action and orbit space from the setting of group actions on locally compact spaces to the context of C^* -dynamical systems [Ri90, Ex99, Ex00, Me01]. This recent revival of interest in the subject is an additional motivation for the general analysis of the continuous Arveson spectrum which we undertake in this work.

Let (α, \mathfrak{A}) be a strongly continuous group of isometries acting on a Banach space \mathfrak{A} and let $(\alpha^*, \mathfrak{A}_*)$ be the transposed action of α on a suitable closed, invariant subspace $\mathfrak{A}_* \subset \mathfrak{A}^*$. In this framework we define the pure-point and continuous spectral subspaces of α and α^* , denoted by \mathfrak{A}_{pp} , \mathfrak{A}_c and $\mathfrak{A}_{*,\text{pp}}$, $\mathfrak{A}_{*,c}$, respectively. Certainly, \mathfrak{A}_{pp} , (resp. $\mathfrak{A}_{*,\text{pp}}$), is spanned by the joint eigenvectors of the generators of α , (resp. α^*). In the absence of orthogonality, our definition of the continuous subspace is motivated by the Ergodic Theorem from the setting of groups of unitaries:

$$\mathfrak{A}_c := \{ A \in \mathfrak{A} \mid \forall_{\substack{q \in \mathbb{R}^d \\ \varphi \in \mathfrak{A}_*}} \lim_{K \nearrow \mathbb{R}^d} \frac{1}{|K|} \int_K e^{-iqx} \varphi(\alpha_x(A)) d^d x = 0 \}. \quad (1.4)$$

The subspace $\mathfrak{A}_{*,c}$ is obtained by exchanging the roles of \mathfrak{A} and \mathfrak{A}_* above. In analogy with the Hilbert space setting,

$$\langle \mathfrak{A}_{*,\text{pp}}, \mathfrak{A}_c \rangle = \langle \mathfrak{A}_{*,c}, \mathfrak{A}_{\text{pp}} \rangle = 0, \quad (1.5)$$

where $\langle \cdot, \cdot \rangle$ denotes the evaluation of functionals from \mathfrak{A}^* on elements of \mathfrak{A} . Exploiting these facts, we find necessary and sufficient conditions for the following decompositions to hold

$$\mathfrak{A} = \mathfrak{A}_{\text{pp}} \oplus \mathfrak{A}_c, \quad (1.6)$$

$$\mathfrak{A}_* = \mathfrak{A}_{*,\text{pp}} \oplus \mathfrak{A}_{*,c}, \quad (1.7)$$

as well as examples, where these equalities fail. More refined analysis of the continuous spectrum requires the choice of some norm dense subspaces $\hat{\mathfrak{A}} \subset \mathfrak{A}$ and $\hat{\mathfrak{A}}_* \subset \mathfrak{A}_*$. By definition, the absolutely continuous part \mathfrak{A}_{ac} is generated by all $A \in \mathfrak{A}_c$ s.t. the Fourier transforms of the corresponding functions $\mathbb{R}^d \ni x \rightarrow \varphi(\alpha_x(A))$ are integrable for all $\varphi \in \hat{\mathfrak{A}}_*$. As for the singular-continuous space, in the Banach space setting the canonical choice is the quotient

$$\mathfrak{A}_{\text{sc}} := \mathfrak{A}_c / \mathfrak{A}_{\text{ac}} \quad (1.8)$$

on which there acts the reduced group of isometries $\underline{\alpha}$, defined naturally on the equivalence classes.

The point-spectrum $\text{Sp}_{\text{pp}} \alpha$ consists of joint eigenvalues of the generators of α . The continuous spectrum of α , denoted by $\text{Sp}_c \alpha$, is defined as the Arveson spectrum of α restricted to the subspace \mathfrak{A}_c . The absolutely continuous spectrum $\text{Sp}_{\text{ac}} \alpha$ is constructed analogously. The singular-continuous part $\text{Sp}_{\text{sc}} \alpha$ is specified as the

Arveson spectrum of $\underline{\alpha}$. The corresponding spectral concepts on the side of α^* are defined in obvious analogy. We remark that in the Hilbert space setting the above spectra and the subspaces \mathfrak{A}_{pp} , \mathfrak{A}_{c} and \mathfrak{A}_{ac} coincide with the standard ones. The space \mathfrak{A}_{sc} is isomorphic, as a Banach space, to the conventional singular-continuous subspace.

We are primarily interested in the case of a group of automorphisms α acting on a C^* -algebra \mathfrak{A} . We assume the existence of a pure state ω_0 on \mathfrak{A} s.t.

$$\ker \omega_0 \subset \mathfrak{A}_{\text{c}}, \quad (1.9)$$

if \mathfrak{A}_* is chosen as the predual of the GNS representation $(\pi, \mathcal{H}, \Omega)$ induced by ω_0 . Then there follow decompositions (1.6) and (1.7) as well as the invariance of ω_0 under the action of α^* . Hence, there acts in \mathcal{H} a strongly-continuous unitary representation U of \mathbb{R}^d which implements α i.e.

$$\pi(\alpha_x(A)) = U(x)\pi(A)U(x)^{-1}, \quad A \in \mathfrak{A}, \quad x \in \mathbb{R}^d. \quad (1.10)$$

We establish new relations between the spectral concepts on the side of α and U . In particular, we find the following *continuity transfer relations*

$$\pi(\mathfrak{A}_{\text{c}})\Omega \subset \mathcal{H}_{\text{c}}, \quad (1.11)$$

$$\pi(\mathfrak{A}_{\text{ac}})\Omega \subset \mathcal{H}_{\text{ac}}, \quad (1.12)$$

akin to the spectrum transfer property (4.23) from the standard Arveson theory. We also verify the inclusions

$$\pm \text{Sp}_{\text{sc}} U \subset \text{Sp}_{\text{sc}} \alpha, \quad (1.13)$$

$$\pm \text{Sp}_{\text{ac}} U \subset \text{Sp}_{\text{ac}} \alpha^*, \quad (1.14)$$

which provide means to estimate the shapes of the spectra introduced above.

This analysis applies, in particular, to the group of spacetime translation automorphisms α in any local, relativistic quantum field theory equipped with a normal vacuum state ω_0 . We obtain from inclusions (1.13), (1.14) that in a theory of Wigner particles the singular-continuous spectrum of α contains the mass hyperboloids of these particles, whereas the multiparticle spectrum contributes to $\text{Sp}_{\text{ac}} \alpha^*$. The subgroup of space translation automorphisms $\beta_{\vec{x}} = \alpha_{(0, \vec{x})}$ allows for a more detailed analysis. Relying on results from [Bu90], we obtain that $\text{Sp}_{\text{ac}} \beta = \text{Sp}_{\text{ac}} \beta^* = \mathbb{R}^s$, whereas $\text{Sp}_{\text{sc}} \beta = \text{Sp}_{\text{sc}} \beta^*$ are either empty or consist of $\{0\}$. These spectra turn out to be empty in theories satisfying certain timelike asymptotic abelianess condition introduced in [BWa92] or complying with a regularity criterion $L^{(1)}$ which restricts the continuous spectrum of α in a neighborhood of zero. We show, following [Dy08.2], that this latter condition implies the existence of particles in theories containing a stress-energy tensor.

This paper is organized as follows: In Section 2 we recall the basics of the Arveson spectral theory, introduce the continuous Arveson spectrum and decompose it into the absolutely continuous and singular-continuous parts. Section 3 focuses

on relations between the spectra of α and α^* . Unitarily implemented groups of automorphisms acting on C^* -algebras are studied in Section 4. In Section 5 we establish general properties of the continuous spectrum of spacetime translation automorphisms valid in any local relativistic quantum field theory admitting a normal vacuum state. In Section 6, which contains some results from author's PhD thesis [Dy08.2], we restrict attention to models complying with the regularity condition $L^{(1)}$. We strengthen our spectral results in this setting and show that such theories describe particles, if they contain a stress-energy tensor. Section 7 summarizes our results and outlines future directions. In the Appendix we consider spectral theory of space translation automorphisms in the absence of normal vacuum states. In particular, we provide examples which violate relations (1.6), (1.7).

2 Continuous spectrum of a group of isometries

In this section we consider a strongly continuous group of isometries $\mathbb{R}^d \ni x \rightarrow \alpha_x$ acting on a Banach space \mathfrak{A} . We choose a subspace $\mathfrak{A}_* \subset \mathfrak{A}^*$ which satisfies the following:

Condition S: The subspace \mathfrak{A}_* is norm closed in \mathfrak{A}^* and invariant under the action of α^* . Moreover, for any $A \in \mathfrak{A}$,

$$\|A\| = \sup_{\varphi \in \mathfrak{A}_{*,1}} |\varphi(A)|. \quad (2.1)$$

The foundations of spectral theory in this setting were laid by Arveson [Ar74, Ar82]. We recall below the familiar concepts of spectral subspaces and the pure-point spectrum. Next, we propose a new notion of continuous Arveson spectrum and decompose it into the absolutely continuous and singular-continuous parts. For this latter purpose we choose a norm dense subspace $\hat{\mathfrak{A}}_* \subset \mathfrak{A}_*$.

For any $\varphi \in \mathfrak{A}_*$ and $A \in \mathfrak{A}$ we consider the Fourier transforms of bounded, continuous functions $\mathbb{R}^d \ni x \rightarrow \varphi(\alpha_x(A))$

$$\varphi(\tilde{A}(p)) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int d^d x e^{-ipx} \varphi(\alpha_x(A)), \quad (2.2)$$

which are tempered distributions. Here px stands for some non-degenerate inner product in \mathbb{R}^d and we adhere to the Fourier transform convention which omits the minus sign in front of px in the case of test-functions. The Arveson spectrum of the group of isometries α is defined as follows

$$\text{Sp } \alpha := \overline{\bigcup_{\substack{A \in \mathfrak{A} \\ \varphi \in \mathfrak{A}_*}} \text{supp } \varphi(\tilde{A}(\cdot))}. \quad (2.3)$$

We also define the Arveson spectrum of an individual element $A \in \mathfrak{A}$ as $\text{Sp}^A \alpha := \text{Sp } \alpha|_X$, where $X = \text{Span}\{\alpha_x(A) \mid x \in \mathbb{R}^d\}^{\text{n-cl}}$ and n-cl denotes the norm closure. (We mention as an aside that for a strongly continuous one-parameter group of

isometries, $\text{Sp } \alpha$ coincides with the operator-theoretic spectrum of the infinitesimal generator $D = \frac{1}{i} \frac{d}{dt} \alpha_t|_{t=0}$ [Ev76, Lo77]). Similarly as in the Hilbert space setting, for any closed set $\Delta \subset \mathbb{R}^d$ one defines the spectral subspace

$$\tilde{\mathfrak{A}}(\Delta) := \{ A \in \mathfrak{A} \mid \forall_{\varphi \in \mathfrak{A}_*} \text{supp } \varphi(\tilde{A}(\cdot)) \subset \Delta \}. \quad (2.4)$$

In particular, for a single point $q \in \mathbb{R}^d$ there holds

$$\tilde{\mathfrak{A}}(\{q\}) = \{ A \in \mathfrak{A} \mid \alpha_x(A) = e^{iqx} A \text{ for all } x \in \mathbb{R}^d \}. \quad (2.5)$$

This leads us to natural definitions of the pure-point subspace and the pure-point spectrum, analogous to those from the Hilbert space setting:

$$\mathfrak{A}_{\text{pp}} := \text{Span}\{ \tilde{\mathfrak{A}}(\{q\}) \mid q \in \text{Sp } \alpha \}^{\text{n-cl}}, \quad (2.6)$$

$$\text{Sp}_{\text{pp}} \alpha := \{ q \in \text{Sp } \alpha \mid \tilde{\mathfrak{A}}(\{q\}) \neq \{0\} \}. \quad (2.7)$$

We note that these objects are independent of the choice of \mathfrak{A}_* , as long as it satisfies Condition S .

The spectral subspaces (2.4) and the pure-point spectrum have been thoroughly studied in the literature. (See [Ar82, Pe] for a review of the former subject and [Ba78, Hu99] for interesting results on the latter). However, there does not seem to exist any generally accepted definition of the continuous Arveson spectrum. Motivated by the Ergodic Theorem from the setting of groups of unitaries acting on Hilbert spaces [RS1], we propose the following norm closed, invariant subspace \mathfrak{A}_c and the corresponding spectrum

$$\mathfrak{A}_c := \{ A \in \mathfrak{A} \mid \forall_{\substack{q \in \mathbb{R}^d \\ \varphi \in \mathfrak{A}_*}} \lim_{K \nearrow \mathbb{R}^d} \frac{1}{|K|} \int_K e^{-iqx} \varphi(\alpha_x(A)) d^d x = 0 \}, \quad (2.8)$$

$$\text{Sp}_c \alpha := \text{Sp } \alpha|_{\mathfrak{A}_c}. \quad (2.9)$$

Here $K \nearrow \mathbb{R}^d$ denotes a family of cuboids, centered at zero, whose edge lengths tend to infinity². In contrast to the pure-point part, the continuous subspace depends on the choice of \mathfrak{A}_* . However, for any such choice $\mathfrak{A}_{\text{pp}} \cap \mathfrak{A}_c = \{0\}$, if $\text{Sp}_{\text{pp}} \alpha$ is a finite set. On the other hand, the equality $\mathfrak{A} = \mathfrak{A}_{\text{pp}} \oplus \mathfrak{A}_c$, expected from the Hilbert space setting, fails in some cases, as we show in the Appendix. Necessary and sufficient conditions for this equality, which we establish in Section 3, will suggest judicious choices of \mathfrak{A}_* .

Let us now introduce more refined spectral concepts which are sensitive to regularity properties of the distributions $\varphi(\tilde{A}(\cdot))$. With the help of the norm dense subspace $\hat{\mathfrak{A}}_*$ in \mathfrak{A}_* we define the absolutely continuous subspace and the corresponding spectrum:

$$\mathfrak{A}_{\text{ac}} := \{ A \in \mathfrak{A}_c \mid \forall_{\varphi \in \hat{\mathfrak{A}}_*} \varphi(\tilde{A}(\cdot)) \in L^1(\mathbb{R}^d, d^d p) \}^{\text{n-cl}}, \quad (2.10)$$

$$\text{Sp}_{\text{ac}} \alpha := \text{Sp } \alpha|_{\mathfrak{A}_{\text{ac}}}. \quad (2.11)$$

²In some cases we impose further restrictions on this family. See the discussion preceding Theorem 5.1.

Let us recall that the singular-continuous part of a Hilbert space is the orthogonal complement of the absolutely continuous subspace in the continuous one. In the case of a Banach space there may not exist any direct sum complement of \mathfrak{A}_{ac} which is invariant under the action of α . (As a matter of fact, there may not exist any direct sum complement at all [Ru]). Therefore, we define the singular continuous *space* as a quotient

$$\mathfrak{A}_{\text{sc}} := \mathfrak{A}_{\text{c}} / \mathfrak{A}_{\text{ac}}. \quad (2.12)$$

We denote the equivalence class of an element $A \in \mathfrak{A}_{\text{c}}$ by $[A]$ and introduce the strongly continuous group of isometries $\mathbb{R}^d \ni x \rightarrow \underline{\alpha}_x$ of \mathfrak{A}_{sc} given by

$$\underline{\alpha}_x[A] := [\alpha_x(A)]. \quad (2.13)$$

We define the singular continuous spectrum as the Arveson spectrum of $\underline{\alpha}$

$$\text{Sp}_{\text{sc}}\alpha := \text{Sp } \underline{\alpha} = \overline{\bigcup_{\substack{A \in \mathfrak{A}_{\text{c}} \\ \varphi \in \mathfrak{A}_{\text{sc}}^*}} \text{supp } \varphi(\widetilde{[A]}(\cdot))}, \quad (2.14)$$

where $\varphi(\widetilde{[A]}(p)) = (2\pi)^{-\frac{d}{2}} \int d^d x e^{-ipx} \varphi(\underline{\alpha}_x([A]))$. Clearly, $\text{Sp}_{\text{sc}}\alpha = \emptyset$, if and only if $\mathfrak{A}_{\text{ac}} = \mathfrak{A}_{\text{c}}$.

To conclude this section, let us consider briefly the case of a group of unitaries α acting on a Hilbert space \mathfrak{A} with $\hat{\mathfrak{A}}_* = \mathfrak{A}_* = \mathfrak{A}^*$. Noting that the distributions $\varphi(\widetilde{A}(\cdot))$ are then just the spectral measures, we obtain that the above concepts of the pure-point, continuous and absolutely continuous spectral subspaces, and the corresponding spectra coincide with the standard ones. Moreover, $\text{Sp}_{\text{sc}}\alpha$ is equal to the conventional singular-continuous spectrum and the space (2.12) is isomorphic, as a Banach space, to the singular-continuous subspace.

3 Spectral analysis of the transposed action

In the setting of groups of unitaries, acting on a Hilbert space, there always holds

$$\mathfrak{A} = \mathfrak{A}_{\text{pp}} \oplus \mathfrak{A}_{\text{c}}. \quad (3.15)$$

However, as shown in the Appendix, in the Banach space setting the above equality fails, if the space \mathfrak{A}_* is excessively large. In order to characterize the choices of \mathfrak{A}_* which entail relation (3.15), we develop the spectral theory for the transposed action α^* . Thus we define the spectra $\text{Sp } \alpha^*$, $\text{Sp}_{\text{pp}} \alpha^*$, $\text{Sp}_{\text{c}} \alpha^*$, as well as the subspaces $\widetilde{\mathfrak{A}}_*(\Delta)$, $\mathfrak{A}_{*,\text{pp}}$, $\mathfrak{A}_{*,\text{c}}$ analogously as in the previous section, by exchanging the roles of \mathfrak{A}_* and \mathfrak{A} . After selecting a norm dense subspace $\hat{\mathfrak{A}} \subset \mathfrak{A}$, we also define $\mathfrak{A}_{*,\text{ac}}$, $\mathfrak{A}_{*,\text{sc}}$ and the corresponding spectra $\text{Sp}_{\text{ac}} \alpha^*$, $\text{Sp}_{\text{sc}} \alpha^*$ ³.

The evaluation of functionals from \mathfrak{A}_* on elements of \mathfrak{A} provides a natural substitute for the scalar product from the Hilbert space setting. We exploit this observation in the following lemma. To stress the analogy with the Hilbert space case,

³If $\underline{\alpha}^*$ does not act norm continuously on $\mathfrak{A}_{*,\text{sc}}$, we refrain from defining $\text{Sp}_{\text{sc}} \alpha^*$. All the other definitions remain meaningful, if the action of α^* is only weakly* continuous.

we denote this evaluation by $\langle \cdot, \cdot \rangle$. Moreover, for any subset $\mathfrak{A}_0 \subset \mathfrak{A}$ we define the annihilator $\mathfrak{A}_0^\perp = \{ \varphi \in \mathfrak{A}_* \mid \forall A \in \mathfrak{A}_0 \varphi(A) = 0 \}$, and analogously for subsets of \mathfrak{A}_* .

Lemma 3.1. *There holds*

- (a) $\langle \tilde{\mathfrak{A}}_*(\Delta), \tilde{\mathfrak{A}}(\Delta') \rangle = 0$, if $\Delta, \Delta' \subset \mathbb{R}^d$ are closed and $\Delta \cap \Delta' = \emptyset$,
- (b) $\langle \mathfrak{A}_{*,\text{pp}}, \mathfrak{A}_c \rangle = 0$ and $\langle \mathfrak{A}_{*,c}, \mathfrak{A}_{\text{pp}} \rangle = 0$.

Proof. As for part (a), let χ_Δ , (resp. $\chi_{\Delta'}$), be a bounded, smooth function on \mathbb{R}^d which is equal to one on Δ , (resp. on Δ'). Moreover, suppose that $\text{supp } \chi_\Delta \cap \text{supp } \chi_{\Delta'} = \emptyset$. Then there holds for any $\varphi \in \tilde{\mathfrak{A}}_*(\Delta)$, $A \in \tilde{\mathfrak{A}}(\Delta')$ and $f \in S(\mathbb{R}^d)$

$$\begin{aligned} \int d^d x \varphi(\alpha_x(A)) f(x) &= \int d^d p \varphi(\tilde{A}(p)) \tilde{f}(p) \\ &= \int d^d p \varphi(\tilde{A}(p)) \tilde{f}(p) \chi_{\Delta'}(p) \chi_\Delta(p) = 0. \end{aligned} \quad (3.16)$$

Therefore $\mathbb{R}^d \ni x \rightarrow \varphi(\alpha_x(A))$ vanishes as a distribution. Since it is a continuous function, it vanishes pointwise.

To prove (b), suppose that $\varphi \in \mathfrak{A}_*$ is an eigenvector i.e. $\alpha_x^* \varphi = e^{iqx} \varphi$ for some $q \in \mathbb{R}^d$. Then there holds for any $A \in \mathfrak{A}_c$

$$\varphi(A) = \frac{1}{|K|} \int_K d^d x e^{-iqx} e^{iqx} \varphi(A) = \frac{1}{|K|} \int_K d^d x e^{-iqx} \varphi(\alpha_x(A)) \xrightarrow{K \nearrow \mathbb{R}^d} 0. \quad (3.17)$$

Since eigenvectors form a total set in $\mathfrak{A}_{*,\text{pp}}$, the proof of the first statement in (b) is complete. The proof of the second statement is analogous. \square

With the help of the above observation we easily obtain the following list of necessary conditions for the decomposition (3.15).

Theorem 3.2. *Suppose that $\mathfrak{A} = \mathfrak{A}_{\text{pp}} \oplus \mathfrak{A}_c$. Then:*

- (a) *For any $q \in \mathbb{R}^d$ there holds $\dim \tilde{\mathfrak{A}}(\{q\}) \geq \dim \tilde{\mathfrak{A}}_*(\{q\})$,*
- (b) $\mathfrak{A}_{\text{pp}}^\perp = \mathfrak{A}_{*,c}$,
- (c) $\text{Sp}_{\text{pp}} \alpha \supset \text{Sp}_{\text{pp}} \alpha^*$ and $\text{Sp}_c \alpha \supset \text{Sp}_c \alpha^*$.

Remark. This result also holds with the roles of \mathfrak{A} and \mathfrak{A}_* exchanged.

Proof. In part (a) it suffices to consider the case $\dim \tilde{\mathfrak{A}}(\{q\}) = m < \infty$. Suppose there exist $m+1$ linearly independent functionals $\{\varphi_k\}_1^{m+1}$ in $\tilde{\mathfrak{A}}_*(\{q\})$. Then, according to Lemma 3.1 (b), their restrictions $\hat{\varphi}_k$ to the subspace

$$\mathring{\mathfrak{A}}_{\text{pp}} := \text{Span}\{ \tilde{\mathfrak{A}}(\{q'\}) \mid q' \in \text{Sp}_{\text{pp}} \alpha \} \quad (3.18)$$

which consists of finite, linear combinations of eigenvectors, still form a linearly independent family. In view of Lemma 3.1 (a),

$$\ker \hat{\varphi}_1 \cap \dots \cap \ker \hat{\varphi}_{m+1} \supset \text{Span}\{ \tilde{\mathfrak{A}}(\{q'\}) \mid q' \in \text{Sp}_{\text{pp}} \alpha, q' \neq q \}, \quad (3.19)$$

which is a contradiction, since the subspace on the r.h.s. of this relation has codimension m in \mathfrak{A}_{pp} .

As for part (b), we note that $\mathfrak{A}_{\text{pp}}^\perp \supset \mathfrak{A}_{*,c}$, by Lemma 3.1 (b). In order to prove the opposite inclusion, we pick $\varphi \in \mathfrak{A}_{\text{pp}}^\perp$ and arbitrary $A \in \mathfrak{A}$. According to our assumption, $A = A_{\text{pp}} + A_c$, where $A_{\text{pp}} \in \mathfrak{A}_{\text{pp}}$ and $A_c \in \mathfrak{A}_c$. There holds for any $q \in \mathbb{R}^d$

$$\frac{1}{|K|} \int_K d^d x e^{-iqx} \varphi(\alpha_x(A)) = \frac{1}{|K|} \int_K d^d x e^{-iqx} \varphi(\alpha_x(A_c)) \xrightarrow{K \nearrow \mathbb{R}^d} 0, \quad (3.20)$$

and therefore $\varphi \in \mathfrak{A}_{*,c}$.

The statement concerning the point spectrum in (c) follows immediately from part (a). As for the continuous spectrum, we note that

$$\text{Sp}_c \alpha = \overline{\bigcup_{\substack{A \in \mathfrak{A}_c \\ \varphi \in \mathfrak{A}_*}} \text{supp } \varphi(\tilde{A}(\cdot))} \supset \overline{\bigcup_{\substack{A \in \mathfrak{A}_c \\ \varphi \in \mathfrak{A}_{*,c}}} \text{supp } \varphi(\tilde{A}(\cdot))} = \text{Sp}_c \alpha^*, \quad (3.21)$$

where the last equality relies on the fact that $\mathfrak{A}_{\text{pp}}^\perp \supset \mathfrak{A}_{*,c}$ and on the assumption. \square

In the Appendix we find counterexamples to relation (3.15). To this end, we exploit part (a) of the above theorem as follows: First, we show that for the group of space translation automorphisms acting on the algebra of observables \mathfrak{A} in quantum field theory there always holds $\dim \mathfrak{A}(\{0\}) = 1$. Next, we note that $\tilde{\mathfrak{A}}_*(\{0\})$ is spanned by the functionals which are invariant under the transposed action. Hence $\mathfrak{A} \neq \mathfrak{A}_{\text{pp}} \oplus \mathfrak{A}_c$, if \mathfrak{A}_* contains more than one vacuum state⁴.

Coming back to the general setting of groups of isometries, we provide sufficient conditions for relation (3.15) to hold. The following theorem accounts for the important role of the point spectrum of α^* in the study of the continuous spectrum of α . Its assumptions have a natural formulation in the framework of automorphism groups of C^* -algebras which we explore in the next section.

Theorem 3.3. *Suppose there exists a functional $\omega_0 \in \mathfrak{A}_*$ s.t. $\ker \omega_0 \subset \mathfrak{A}_c$ and a non-zero element $I \in \tilde{\mathfrak{A}}(\{q\})$ for some $q \in \mathbb{R}^d$. Then:*

- (a) $\mathfrak{A} = \mathfrak{A}_{\text{pp}} \oplus \mathfrak{A}_c$, where $\mathfrak{A}_{\text{pp}} = \text{Span } \{I\}$, $\mathfrak{A}_c = \ker \omega_0$,
- (b) $\omega_0 \in \tilde{\mathfrak{A}}_*(\{q\})$,
- (c) $\mathfrak{A}_* = \mathfrak{A}_{*,\text{pp}} \oplus \mathfrak{A}_{*,c}$, where $\mathfrak{A}_{*,\text{pp}} = \text{Span } \{\omega_0\}$, $\mathfrak{A}_{*,c} = \ker I$.

Moreover, $\text{Sp}_c \alpha = \text{Sp}_c \alpha^*$.

Proof. By assumption, \mathfrak{A}_c is a subspace of codimension at most one in \mathfrak{A} . Since $\mathfrak{A}_{\text{pp}} \supset \text{Span } \{I\}$, part (a) follows. Hence any $A \in \mathfrak{A}$ can be expressed as $A = cI + A_c$, where $A_c \in \mathfrak{A}_c$. Since \mathfrak{A}_c is invariant under the action of automorphisms, there holds $\alpha_x^* \omega_0(A) = c \omega_0(\alpha_x(I)) = e^{iqx} \omega_0(A)$, which entails (b). From Theorem 3.2 (b) we obtain that $\mathfrak{A}_{*,c} = \ker I$. Since it is a subspace of codimension one, part (c) follows from part (b). The last statement is a consequence of Theorem 3.2 (c) and parts (a), (c) of the present theorem. \square

⁴That is a translationally invariant state on the algebra of observables s.t. the relativistic spectrum condition holds in its GNS representation (cf. Section 5).

4 Spectral analysis of a group of automorphisms

In this section we consider a group of automorphisms α acting on a C^* -algebra \mathfrak{A} containing a unity I . We assume that there exists a pure state ω_0 on \mathfrak{A} which satisfies

$$\ker \omega_0 \subset \mathfrak{A}_c, \quad (4.22)$$

where \mathfrak{A}_* , entering the definition of \mathfrak{A}_c , is chosen as the predual of the GNS representation $(\pi, \mathcal{H}, \Omega)$ induced by the state ω_0 . (As we will see in Section 5, in the case of spacetime translation automorphisms in QFT any pure vacuum state satisfies this inclusion as a consequence of locality). On the subspace $\hat{\mathfrak{A}}_* \subset \mathfrak{A}_*$ we impose the following condition:

Condition \hat{S} : The subspace $\hat{\mathfrak{A}}_*$ is self-adjoint⁵, norm dense in \mathfrak{A}_* and contains all the functionals of the form $(\Psi|\pi(\cdot)\Omega)$, where Ψ belongs to some dense subspace in \mathcal{H} .

Having specified the framework, we proceed to the spectral analysis of α . Exploiting Theorem 3.3 and the fact that \mathfrak{A} is unital, we obtain:

Theorem 4.1. *Let \mathfrak{A} , ω_0 and \mathfrak{A}_* be specified as above. Then:*

- (a) $\mathfrak{A} = \mathfrak{A}_{pp} \oplus \mathfrak{A}_c$, where $\mathfrak{A}_{pp} = \text{Span}\{I\}$, $\mathfrak{A}_c = \ker \omega_0$,
- (b) $\omega_0 \in \tilde{\mathfrak{A}}_*(\{0\})$,
- (c) $\mathfrak{A}_* = \mathfrak{A}_{*,pp} \oplus \mathfrak{A}_{*,c}$, where $\mathfrak{A}_{*,pp} = \text{Span}\{\omega_0\}$, $\mathfrak{A}_{*,c} = \ker I$.

Moreover, $\text{Sp}_c \alpha = \text{Sp}_c \alpha^*$.

In view of part (b) of the above theorem, the state ω_0 is invariant under the action of α^* . Hence there exists a strongly continuous group of unitaries $\mathbb{R}^d \ni x \rightarrow U(x)$, acting on \mathcal{H} , s.t. $\pi(\alpha_x(A)) = U(x)\pi(A)U(x)^{-1}$ and the vector $\Omega \in \mathcal{H}$ is invariant under its action. We denote by $\tilde{\mathcal{H}}(\Delta)$, \mathcal{H}_{pp} , \mathcal{H}_c , \mathcal{H}_{ac} , \mathcal{H}_{sc} the spectral subspaces of \mathcal{H} w.r.t. the action of U . One of the central problems in the present setting is to find relations between the Arveson spectrum of α and the spectrum of the implementing group of unitaries. An important and well known property is the *spectrum transfer*: If $A \in \tilde{\mathfrak{A}}(\Delta_1)$ then

$$\pi(A)\tilde{\mathcal{H}}(\Delta_2) \subset \tilde{\mathcal{H}}(\overline{\Delta_1 + \Delta_2}), \quad (4.23)$$

where $\Delta_1, \Delta_2 \subset \mathbb{R}^d$ are closed sets [Ar82]. It turns out that similar properties, which can be called *continuity transfer* relations, hold at the level of more detailed spectral theory:

Proposition 4.2. *Let \mathfrak{A} , ω_0 and \mathfrak{A}_* be specified as above and suppose that Condition \hat{S} holds. Then:*

- (a) $\mathcal{H}_{pp} = \text{Span}\{\Omega\}$,

⁵That is if $\varphi \in \hat{\mathfrak{A}}_*$ then $\overline{\varphi} \in \hat{\mathfrak{A}}_*$, where $\overline{\varphi}(A) = \overline{\varphi(A^*)}$, $A \in \mathfrak{A}$.

$$(b) \mathcal{H}_c = \{ \pi(\mathfrak{A}_c)\Omega \}^{\text{n-cl}},$$

$$(c) \mathcal{H}_{ac} \supset \pi(\mathfrak{A}_{ac})\Omega.$$

Proof. First, we show that $\pi(\mathfrak{A}_c)\Omega \subset \mathcal{H}_c$: Let $A \in \mathfrak{A}_c$ and $\Psi \in \mathcal{H}$ be arbitrary. Then for any $q \in \mathbb{R}^d$

$$\frac{1}{|K|} \int_K d^d x e^{-iqx} (\Psi | U(x) \pi(A) \Omega) = \frac{1}{|K|} \int_K d^d x e^{-iqx} \varphi(\alpha_x(A)) \xrightarrow{K \nearrow \mathbb{R}^d} 0, \quad (4.24)$$

where in the last step we made use of the fact that $\varphi(\cdot) = (\Psi | \pi(\cdot) \Omega) \in \mathfrak{A}_*$. Thus, by the Ergodic Theorem, $\pi(A)\Omega \in \mathcal{H}_c$.

Next, we verify that $\{ \pi(\mathfrak{A}_c)\Omega \}^{\text{n-cl}} = \{\Omega\}^\perp$. Suppose that $\Psi \in \mathcal{H}$ and $(\Psi | \Omega) = 0$. By cyclicity of Ω , there exists a sequence $\{A_n\}_{n \in \mathbb{N}}$ of elements of \mathfrak{A} s.t. $\|\Psi - \pi(A_n)\Omega\| \rightarrow 0$. Then $B_n := A_n - \omega_0(A_n)I$ is a sequence of elements of \mathfrak{A}_c which satisfies $\|\Psi - \pi(B_n)\Omega\| \rightarrow 0$. This completes the proof of (a) and (b).

To prove (c), let $A \in \mathfrak{A}_{ac}$ and let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of elements of \mathfrak{A}_{ac} which converges to A in norm and s.t.

$$\int d^d p |\varphi(\tilde{A}_n(p))| < \infty \quad (4.25)$$

for all $n \in \mathbb{N}$ and $\varphi \in \hat{\mathfrak{A}}_*$. Then, for any Ψ from the dense set appearing in Condition \hat{S} ,

$$\int d^d p |(\Psi | \widetilde{U(\cdot) \pi(A_n) \Omega})(p)| = \int d^d p |(\Psi | \pi(\tilde{A}_n(p)) \Omega)| < \infty. \quad (4.26)$$

Hence the vectors $\pi(A_n)\Omega$ belong to \mathcal{H}_{ac} and so does their norm limit $\pi(A)\Omega$. \square

After this preparation we establish relations between the spectra of U and α . We use these facts in the next section, in a quantum field theoretic context.

Theorem 4.3. *Under the assumptions of Proposition 4.2 there hold the relations:*

$$(a) \text{Sp}_{pp} U = \text{Sp}_{pp} \alpha = \text{Sp}_{pp} \alpha^* = \{0\},$$

$$(b) \text{Sp}_c U - \text{Sp}_c U \subset \text{Sp}_c \alpha,$$

$$(c) \pm \text{Sp}_{ac} U \subset \text{Sp}_{ac} \alpha^*,$$

$$(d) \pm \text{Sp}_{sc} U \subset \text{Sp}_{sc} \alpha.$$

Proof. Part (a) follows immediately from Proposition 4.2 (a) and Theorem 4.1. To prove (b), let $q = p_1 - p_2$, where $p_1, p_2 \in \text{Sp}_c U$. We choose an open neighborhood V_q of q and bounded neighborhoods Δ_1, Δ_2 of p_1 and p_2 , respectively, s.t. $\overline{\Delta_1 - \Delta_2} \subset V_q$. We pick $\Psi_1 \in \text{Ran } P(\Delta_1)$ and $\Psi_2 \in \text{Ran } P(\Delta_2)$. Since the continuous spectrum of U is a subset of the essential spectrum, we can find such Ψ_1, Ψ_2 that $(\Psi_1 | \Psi_2) = 0$. By purity of the state ω_0 , $\pi(\mathfrak{A})$ acts irreducibly on \mathcal{H} and we can find such $A \in \mathfrak{A}$ that

$$(\Psi_1 | \pi(A) \Psi_2) \neq 0. \quad (4.27)$$

Replacing A with $A - \omega_0(A)I$, if necessary, we can assume that $A \in \mathfrak{A}_c$. Now we choose $f \in S(\mathbb{R}^d)$ s.t. $\tilde{f}(p) = (2\pi)^{-d/2}$ for $p \in \overline{\Delta_1 - \Delta_2}$ and $\tilde{f}(p) = 0$ for p outside of V_q . Making use of the fact that the support of the distribution $(\Psi_1|\pi(\tilde{A}(\cdot))\Psi_2)$ is contained in $\overline{\Delta_1 - \Delta_2}$, which follows from relation (4.23), we obtain

$$\begin{aligned} \int d^d x f(x)(\Psi_1|\pi(\alpha_x(A))\Psi_2) &= \int d^d p \tilde{f}(p)(\Psi_1|\pi(\tilde{A}(p))\Psi_2) \\ &= (\Psi_1|\pi(A)\Psi_2) \neq 0, \end{aligned} \quad (4.28)$$

which implies that $q \in \text{Sp}_c \alpha$.

As for (c), let $q \in \text{Sp}_{ac} U$ and let V_q be any open neighborhood of q . Then there exists $f \in S(\mathbb{R}^d)$ s.t. $\text{supp } \tilde{f} \subset V_q$ and $\Psi_f := \int d^d x f(x)U(x)\Psi \neq 0$ for some $\Psi \in \mathcal{H}_{ac}$. We note that the functional $\varphi_\Psi(\cdot) := (\Psi|\pi(\cdot)\Omega)$ belongs to $\mathfrak{A}_{*,ac}$, since for any $A \in \mathfrak{A}$

$$\int d^d p |\varphi_\Psi(\tilde{A}(p))| = \int d^d p |(\Psi|U(\cdot)\pi(A)\Omega)(p)| < \infty, \quad (4.29)$$

where in the last step we made use of the fact that $\Psi \in \mathcal{H}_{ac}$. Next, since $\pi(\mathfrak{A})\Omega$ is dense in \mathcal{H} , we find such $B \in \mathfrak{A}$ that $(\Psi_f|\pi(B)\Omega) \neq 0$. Hence

$$\int d^d x \bar{f}_-(x) \alpha_x^* \varphi_\Psi(B) \neq 0, \quad (4.30)$$

where $f_-(x) = f(-x)$. Since $\text{supp } \tilde{f}_- \subset V_q$, we obtain that $q \in \text{Sp}_{ac} \alpha^*$. To show that $-q \in \text{Sp}_{ac} \alpha^*$, we repeat the argument using the functional $\bar{\varphi}_\Psi(\cdot) = (\Omega|\pi(\cdot)\Psi)$ instead of φ_Ψ .

To prove (d), let $q \in \text{Sp}_{sc} U$ and let V_q be an open neighborhood of q . Then there exists $f \in S(\mathbb{R}^d)$ s.t. $\text{supp } \tilde{f} \subset V_q$ and $\Phi_f := \int d^d x f(x)U(x)\Phi \neq 0$ for some $\Phi \in \mathcal{H}_{sc}$. By Proposition 4.2 (b), $\pi(\mathfrak{A}_c)\Omega$ is dense in \mathcal{H}_c , so we can find $A \in \mathfrak{A}_c$ s.t.

$$(\Phi_f|\pi(A)\Omega) \neq 0. \quad (4.31)$$

We note that the functional $\varphi_\Phi(\cdot) = (\Phi|\pi(\cdot)\Omega)$ is an element of \mathfrak{A}_c^* which, by Proposition 4.2 (c), contains \mathfrak{A}_{ac} in its kernel. Therefore, φ_Φ induces a well defined, bounded functional $\underline{\varphi}_\Phi$ on $\mathfrak{A}_{sc} = \mathfrak{A}_c/\mathfrak{A}_{ac}$ s.t. $\underline{\varphi}_\Phi([B]) = \varphi_\Phi(B)$ for $B \in \mathfrak{A}_c$. By relation (4.31),

$$\int d^d x \bar{f}_-(x) \underline{\varphi}_\Phi(\alpha_x[A]) \neq 0, \quad (4.32)$$

which proves that $q \in \text{Sp}_{sc} \alpha$. To show that $-q \in \text{Sp}_{sc} \alpha$, we repeat the argument using the functional $\bar{\varphi}_\Phi(\cdot) = (\Omega|\pi(\cdot)\Phi)$ instead of φ_Φ . \square

5 Spectral theory of automorphism groups in QFT

In this section we analyze the spectrum of the group of spacetime translation automorphisms acting on the algebra of observables in quantum field theory. To keep our

investigation general, we rely on the Haag-Kastler framework of algebraic quantum field theory [Ha]: The theory is based on a net $\mathcal{O} \rightarrow \mathfrak{A}(\mathcal{O}) \subset B(\mathcal{H})$ of unital C^* -algebras, attached to open, bounded regions of spacetime $\mathcal{O} \subset \mathbb{R}^{s+1}$, which satisfies isotony and locality. The $*$ -algebra of local operators is given by

$$\mathfrak{A}_{\text{loc}} := \bigcup_{\mathcal{O} \subset \mathbb{R}^{s+1}} \mathfrak{A}(\mathcal{O}) \quad (5.1)$$

and its norm closure \mathfrak{A} is irreducibly represented on the infinitely dimensional Hilbert space \mathcal{H} . Moreover, \mathcal{H} carries a strongly continuous unitary representation of translations $\mathbb{R}^{s+1} \ni x \rightarrow U(x)$ which satisfies the relativistic spectral condition i.e. the joint spectrum of the generators H, P_1, \dots, P_s is contained in the closed forward lightcone \overline{V}_+ . It is also assumed that the translation automorphisms $\alpha_x(\cdot) = U(x) \cdot U(x)^*$ act geometrically on the net i.e.

$$\alpha_x(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\mathcal{O} + x), \quad (5.2)$$

and are strongly continuous, that is any function $\mathbb{R}^{s+1} \ni x \rightarrow \alpha_x(A)$, $A \in \mathfrak{A}$, is continuous in the norm topology of \mathfrak{A} . Suppose that the Hilbert space contains a vacuum vector Ω , which is invariant under the action of U . Then we set $\omega_0(\cdot) = (\Omega | \cdot | \Omega)$ and say that the theory admits a normal vacuum state ω_0 . This state belongs to the general class of functionals of bounded energy, defined as follows: Let P_E be the spectral projection of H (the Hamiltonian) on the subspace spanned by vectors of energy lower than or equal to E . We identify $B(\mathcal{H})_*$ with the space of trace-class operators on \mathcal{H} and denote by S_E the set of states from $P_E B(\mathcal{H})_* P_E$. The following norm dense subspace of $B(\mathcal{H})_*$

$$B(\mathcal{H})_{*,\text{bd}} := \text{Span}\left\{\bigcup_{E \geq 0} S_E\right\} \quad (5.3)$$

is called the space of functionals of bounded energy.

It is easy to incorporate the triple $(\alpha, \mathfrak{A}, \omega_0)$, introduced above, into the algebraic setting of the previous section: We note that the GNS representation of \mathfrak{A} , induced by ω_0 , coincides with the defining representation, up to unitary equivalence. We set

$$\mathfrak{A}_* = B(\mathcal{H})_*, \quad \hat{\mathfrak{A}}_* = B(\mathcal{H})_{*,\text{bd}}, \quad \hat{\mathfrak{A}} = \mathfrak{A}_{\text{loc}} \quad (5.4)$$

and check that Condition \hat{S} is satisfied. It remains to verify that there holds the key assumption (4.22) i.e. $\ker \omega_0 \subset \mathfrak{A}_c$. To this end, we choose the family of sets $K \nearrow \mathbb{R}^{s+1}$, appearing in definition (2.8), so as to exploit the local structure of the theory. In the case of spacetime translations we set $K_L = [-L^\varepsilon, L^\varepsilon] \times [-L, L]^{\times s}$ for some $0 < \varepsilon < 1$ and any $L \in \mathbb{R}_+$, while for the subgroup of space translations we choose $K'_L = [-L, L]^{\times s}$. Clearly, $K_L \nearrow \mathbb{R}^{s+1}$ and $K'_L \nearrow \mathbb{R}^s$ as $L \rightarrow \infty$. Since we work with the Minkowski space \mathbb{R}^{s+1} , we set $px = p_0 x_0 - \vec{p}\vec{x}$, $\vec{p}\vec{x} = \sum_{j=1}^s p_j x_j$, in definitions (2.2) and (2.8). Accordingly, $px = p_0 x_0$ for the time axis and $px = -\vec{p}\vec{x}$ for the spacelike hyperplane \mathbb{R}^s . After this preparation, we obtain:

Theorem 5.1. *Let $\mathbb{R}^{s+1} \ni x \rightarrow \alpha_x$ be the group of spacetime translation automorphisms acting on the algebra of observables \mathfrak{A} in a quantum field theory admitting a normal vacuum state ω_0 and let $\mathfrak{A}_* = B(\mathcal{H})_*$. Then $\ker \omega_0 \subset \mathfrak{A}_c$ and, consequently,*

- (a) $\mathfrak{A} = \mathfrak{A}_{\text{pp}} \oplus \mathfrak{A}_c$, where $\mathfrak{A}_{\text{pp}} = \text{Span}\{I\}$, $\mathfrak{A}_c = \ker \omega_0$,
- (b) $\mathfrak{A}_* = \mathfrak{A}_{*,\text{pp}} \oplus \mathfrak{A}_{*,c}$, where $\mathfrak{A}_{*,\text{pp}} = \text{Span}\{\omega_0\}$, $\mathfrak{A}_{*,c} = \ker I$,
- (c) $\text{Sp}_c \alpha = \text{Sp}_c \alpha^*$.

The above statements are also true for the subgroup $\mathbb{R}^s \ni \vec{x} \rightarrow \beta_{\vec{x}} := \alpha_{(0,\vec{x})}$ of space translation automorphisms.

Proof. To verify that $\ker \omega_0 \subset \mathfrak{A}_c$, we have to show that for any $A \in \ker \omega_0$ and $q \in \mathbb{R}^{s+1}$

$$\text{w}^* - \lim_{L \rightarrow \infty} \frac{1}{|K_L|} \int_{K_L} d^{s+1}x e^{-iqx} \alpha_x(A) = 0. \quad (5.5)$$

This can be proven by a rather standard argument: We note that for any normal functional $\varphi \in B(\mathcal{H})_*$ the function $\mathbb{R}^{s+1} \ni x \rightarrow \alpha_x^* \varphi$ is continuous w.r.t. the norm topology in $B(\mathcal{H})_*$ so the following Bochner integrals

$$\varphi_L = \frac{1}{|K_L|} \int_{K_L} d^{s+1}x e^{-iqx} \alpha_x^* \varphi \quad (5.6)$$

define functionals from $B(\mathcal{H})_*$. Now we fix a state $\omega \in B(\mathcal{H})_*$ and obtain, from the Banach-Alaoglu theorem, a net $\{K_{L_\beta} \subset \mathbb{R}^{s+1} \mid \beta \in \mathbb{I}\}$ and a functional $\omega^\mathbb{I} \in \mathfrak{A}^*$ s.t.

$$\text{w}^* - \lim_{\beta} \omega_{L_\beta} = \omega^\mathbb{I}. \quad (5.7)$$

Next, for any $A \in \mathfrak{A}$, we consider the following net of elements from $B(\mathcal{H})$

$$P_L(A) := \frac{1}{|K_L|} \int_{K_L} d^{s+1}x e^{-iqx} \alpha_x(A). \quad (5.8)$$

By locality and the slow growth of the timelike dimension of K_L , the net $\{P_L(A)\}_{L>0}$ satisfies

$$\lim_{L \rightarrow \infty} \|[P_L(A), B]\| = 0 \quad (5.9)$$

for any $B \in \mathfrak{A}$. Therefore, all its limit points w.r.t. the weak* topology on $B(\mathcal{H})$ are multiples of the identity by the assumed irreducibility of \mathfrak{A} . It follows that for any $\varphi \in B(\mathcal{H})_*$, $A \in \mathfrak{A}$

$$\lim_{L \rightarrow \infty} (\varphi_L(A) - \omega_L(A) \varphi(I)) = 0. \quad (5.10)$$

Consequently

$$\text{w}^* - \lim_{\beta} \frac{1}{|K_{L_\beta}|} \int_{K_{L_\beta}} d^{s+1}x e^{-iqx} \alpha_x(A) = \omega^\mathbb{I}(A)I. \quad (5.11)$$

By evaluating this relation on the translationally invariant state ω_0 , we obtain that $\omega^{\mathbb{I}} = \omega_0$ for $q = 0$ and $\omega^{\mathbb{I}} = 0$ otherwise, which entails equality (5.5). Now parts (a), (b) and (c) follow from Theorem 4.1. By an obvious modification of the above argument we obtain the statement concerning the subgroup of space translations. \square

Proceeding to more detailed analysis of the spectrum of α , we obtain, with the help of Theorem 4.3, the following facts:

Theorem 5.2. *Let $\mathbb{R}^{s+1} \ni x \rightarrow \alpha_x$ be the group of spacetime translation automorphisms acting on the algebra of observables \mathfrak{A} in a quantum field theory admitting a normal vacuum state ω_0 . Let $m > 0$ and suppose that $\text{Sp} U = \{0\} \cup \mathfrak{h}_m \cup \mathfrak{g}_m$, where $\mathfrak{h}_m = \{p \in \mathbb{R}^{s+1} \mid p^2 = m^2, p^0 > 0\}$ is the mass hyperboloid and $\mathfrak{g}_m = \{p \in \mathbb{R}^{s+1} \mid p^2 \geq (2m)^2, p^0 > 0\}$ is the multiparticle spectrum. Then there holds:*

- (a) $\text{Sp}_{\text{pp}} \alpha = \text{Sp}_{\text{pp}} \alpha^* = \{0\}$,
- (b) $\text{Sp}_{\text{c}} \alpha = \text{Sp}_{\text{c}} \alpha^* = \mathbb{R}^{s+1}$,
- (c) $\text{Sp}_{\text{ac}} \alpha^* \supset \pm \mathfrak{g}_m$,
- (d) $\text{Sp}_{\text{sc}} \alpha \supset \pm \mathfrak{h}_m$.

Here the subspaces \mathfrak{A}_* , $\hat{\mathfrak{A}}_*$ and $\hat{\mathfrak{A}}$ are chosen as in (5.4).

Proof. Parts (a) and (b) follow directly from the corresponding statements in Theorem 4.3 and from Theorem 5.1 (c). Part (c) is a consequence of Theorem 4.3 (c) under the premise that $\mathfrak{g}_m \subset \text{Sp}_{\text{ac}} U$. To prove this fact, we proceed as follows: Let \mathcal{H}_m be the spectral subspace corresponding to \mathfrak{h}_m and let $\mathcal{H}_0 = \Gamma(\mathcal{H}_m)$ be the symmetric Fock space over \mathcal{H}_m . Moreover, let U_m be the restriction of U to \mathcal{H}_m and let $U_0 = \Gamma(U_m)$ be its second quantization acting on \mathcal{H}_0 . Then, by the Haag-Ruelle scattering theory [BF82, Ar, Dy05], there exists the isometric wave-operator $\Omega_+ : \mathcal{H}_0 \rightarrow \mathcal{H}$ which satisfies

$$\Omega_+ U_0(x) = U(x) \Omega_+, \quad x \in \mathbb{R}^{s+1}. \quad (5.12)$$

It is a simple exercise to show that $\mathfrak{g}_m = \text{Sp}_{\text{ac}} U_0$. Hence, if $q \in \mathfrak{g}_m$, then for any open neighborhood V_q of q there exists $f \in S(\mathbb{R}^{s+1})$ s.t. $\text{supp } \tilde{f} \subset V_q$ and

$$\int d^{s+1}x U_0(x) f(x) \Psi_0 \neq 0 \quad (5.13)$$

for some $\Psi_0 \in \mathcal{H}_{0,\text{ac}}$. Thus, making use of property (5.12) and the fact that Ω_+ is an isometry, we obtain that $\Omega_+ \Psi_0 \in \mathcal{H}_{\text{ac}}$ and

$$\int d^{s+1}x U(x) f(x) \Omega_+ \Psi_0 \neq 0, \quad (5.14)$$

which proves that $q \in \text{Sp}_{\text{ac}} U$.

To justify (d), we note that \mathfrak{h}_m is a set of Lebesgue measure zero in \mathbb{R}^{s+1} . Hence, if $q \in \mathfrak{h}_m$, then it either belongs to the singular-continuous spectrum or to the pure-point spectrum of U . The latter possibility is excluded by Proposition 4.2 (a) (or Lemma 3.2.5 of [Ha]). Now the statement follows from Theorem 4.3 (d). \square

This theorem exhibits an interplay between the spectral properties of α and the particle aspects of quantum field theory: The mass hyperboloids of Wigner particles contribute to the singular-continuous spectrum of α . Thereby, this result provides a large class of physically relevant examples of automorphism groups with non-empty singular-continuous spectrum. However, it leaves open the question of non-triviality of $\text{Sp}_{\text{ac}}\alpha$. Since we do not have sufficient control over the full group of spacetime translations, we will study this problem in the case of the subgroup $\mathbb{R}^s \ni \vec{x} \rightarrow \beta_{\vec{x}} := \alpha_{(0, \vec{x})}$ of space translation automorphisms.

Theorem 5.1 fixes the decomposition of \mathfrak{A} and \mathfrak{A}_* into the pure-point and continuous parts w.r.t. the action of β . To facilitate further analysis, we introduce the following subspaces

$$\hat{\mathfrak{A}}_c := \{ A \in \hat{\mathfrak{A}} \mid \omega_0(A) = 0 \}, \quad (5.15)$$

$$\hat{\mathfrak{A}}_{*,c} := \{ \varphi \in \hat{\mathfrak{A}}_* \mid \varphi(I) = 0 \} \quad (5.16)$$

which are norm dense in \mathfrak{A}_c and $\mathfrak{A}_{*,c}$, respectively. Our study of the absolutely continuous and singular-continuous spectrum of β is based on two ingredients: First of them is the following fact, mentioned in [Bu90].

Lemma 5.3. *For any non-zero $A \in \hat{\mathfrak{A}}_c$*

$$\overline{\bigcup_{\varphi \in \hat{\mathfrak{A}}_{*,c}} \text{supp } \varphi(\tilde{A}(\cdot))} = \mathbb{R}^s, \quad (5.17)$$

where the Fourier transform is taken w.r.t. the group of space translations β .

Proof. Let X be the set on the l.h.s. of relation (5.17) and X_0 its counterpart with $\hat{\mathfrak{A}}_{*,c}$ replaced with its norm closure $\mathfrak{A}_{*,c}$. First, we prove that $X_0 = \mathbb{R}^s$. In fact, suppose that the complement of X_0 is a non-empty (open) set. Then, for any operator $B \in \hat{\mathfrak{A}}$, and $\varphi \in B(\mathcal{H})_*$, the functional $\varphi(\cdot B) - \varphi(B \cdot)$ is in $\mathfrak{A}_{*,c}$. Hence the following analytic function of $\vec{p} \in \mathbb{R}^s$

$$\varphi([\tilde{A}(\vec{p}), B]) = \frac{1}{(2\pi)^{s/2}} \int d^s x e^{i\vec{p}\vec{x}} \varphi([\beta_{\vec{x}}(A), B]) \quad (5.18)$$

is identically equal to zero. Thus A belongs to the commutant of \mathfrak{A} which consists of multiples of the identity. Since $\omega_0(A) = 0$, we obtain that $A = 0$, which is a contradiction.

Now suppose that X has a non-empty (open) complement O in \mathbb{R}^s . Then, for any $f \in S(\mathbb{R}^s)$ s.t. $\text{supp } \tilde{f} \subset O$ and for any $\varphi \in \hat{\mathfrak{A}}_{*,c}$, $\varphi(A(f)) = 0$ holds, where

$$A(f) := \int d^s x f(\vec{x}) \beta_{\vec{x}}(A) \in \mathfrak{A}. \quad (5.19)$$

Since $\hat{\mathfrak{A}}_{*,c}$ is norm dense in $\mathfrak{A}_{*,c}$, the same holds for $\varphi \in \mathfrak{A}_{*,c}$, contradicting the fact that $X_0 = \mathbb{R}^s$. \square

The second ingredient is the following estimate, due to Buchholz [Bu90],

$$\sup_{\omega \in S_E} \int d^s p |\vec{p}|^{s+1+\varepsilon} |\omega(\tilde{A}(\vec{p}))|^2 < \infty, \quad (5.20)$$

valid for any $E \geq 0$, any local observable $A \in \mathfrak{A}_{\text{loc}}$ and $\varepsilon > 0$. With these two facts at hand, we are ready to analyze the spectrum of β .

Theorem 5.4. *Let $\mathbb{R}^s \ni \vec{x} \rightarrow \beta_{\vec{x}}$ be the group of space translation automorphisms acting on the algebra of observables \mathfrak{A} in a quantum field theory admitting a normal vacuum state ω_0 . Then there holds:*

- (a) $\text{Sp}_{\text{pp}}\beta = \text{Sp}_{\text{pp}}\beta^* = \{0\}$,
- (b) $\text{Sp}_{\text{ac}}\beta = \text{Sp}_{\text{ac}}\beta^* = \mathbb{R}^s$,
- (c) $\text{Sp}_{\text{sc}}\beta \subset \{0\}$, $\text{Sp}_{\text{sc}}\beta^* \subset \{0\}$.

The subspaces \mathfrak{A}_* , $\hat{\mathfrak{A}}_*$ and $\hat{\mathfrak{A}}$ are given by (5.4).

Proof. Part (a) follows directly from Theorem 5.1. To prove (b) and (c), we proceed as follows: For any function $f \in C_0^\infty(\mathbb{R}^s)$ and $n \in \mathbb{N}$ we introduce $f_n \in C_0^\infty(\mathbb{R}^s)$ given by $\tilde{f}_n(\vec{p}) = \tilde{f}(\vec{p})|\vec{p}|^{2n}$. Next, for any $A \in \hat{\mathfrak{A}}_c$ and $\varphi \in \hat{\mathfrak{A}}_{*,c}$ we set

$$A(f_n) := \int d^s x f_n(\vec{x}) \beta_{\vec{x}}(A), \quad (5.21)$$

$$\varphi_{f_n} := \int d^s x f_n(\vec{x}) \beta_{\vec{x}}^* \varphi. \quad (5.22)$$

We note that $A(f_n) \in \hat{\mathfrak{A}}_c$, since each local algebra $\mathfrak{A}(\mathcal{O})$ is a norm closed subspace of \mathfrak{A} and the action of β is strongly continuous. Similarly, $\varphi_{f_n} \in \hat{\mathfrak{A}}_{*,c}$, since β^* acts strongly continuously on $B(\mathcal{H})_*$ and each $\varphi \in \hat{\mathfrak{A}}_{*,c}$ belongs to the closed subspace $\text{Span } S_E \subset B(\mathcal{H})_*$ for E sufficiently large. Setting $4n > s + 1$ and noting that $\widetilde{A(f_n)}(\vec{p}) = (2\pi)^{s/2} \tilde{f}_n(\vec{p}) \tilde{A}(\vec{p})$, we obtain from estimate (5.20)

$$\forall_{\substack{\varphi \in \hat{\mathfrak{A}}_{*,c} \\ A \in \hat{\mathfrak{A}}_c}} \int d^s p |\varphi(\widetilde{A(f_n)}(\vec{p}))|^2 < \infty. \quad (5.23)$$

Hence, recalling definition (2.10) and noting that the distributions $\varphi(\widetilde{A(f_n)}(\cdot))$ are compactly supported, we conclude that $A(f_n) \in \mathfrak{A}_{\text{ac}}$ and $\varphi_{f_n} \in \mathfrak{A}_{*,\text{ac}}$. Now we show that some of these elements are different from zero: Clearly, for any non-zero $A \in \hat{\mathfrak{A}}_c$ one can choose such $f \in C_0^\infty(\mathbb{R}^s)$ that $A(f) \neq 0$. Thus we conclude from Lemma 5.3 that $\text{supp } \varphi(\widetilde{A(f)}(\cdot))$ contains $\vec{p} \neq 0$ for some $\varphi \in \hat{\mathfrak{A}}_{*,c}$. Hence, $\varphi(A(f_n)) \neq 0$ or,

equivalently, $\varphi_{f_n}(A) \neq 0$. Now part (b) of the theorem follows from Lemma 5.3 and the inclusions

$$\overline{\bigcup_{\substack{A \in \hat{\mathfrak{A}}_c \\ \varphi \in \mathfrak{A}_*}} \text{supp } \varphi(\widetilde{A(f_n)(\cdot)})} \subset \text{Sp}_{ac}\beta, \quad (5.24)$$

$$\overline{\bigcup_{\substack{A \in \mathfrak{A} \\ \varphi \in \hat{\mathfrak{A}}_{*,c}}} \text{supp } \varphi_{f_n}(\widetilde{A(\cdot)})} \subset \text{Sp}_{ac}\beta^*. \quad (5.25)$$

To verify the first statement in part (c), we have to show that for any $A \in \mathfrak{A}_c$ the corresponding element $[A] \in \mathfrak{A}_{sc} = \mathfrak{A}_c / \mathfrak{A}_{ac}$ satisfies $[A](f) = [A(f)] = 0$ for any $f \in S(\mathbb{R}^s)$ s.t. $\text{supp } \tilde{f} \cap \{0\} = \emptyset$. To this end, we pick a sequence $\{A_m\}_{m \in \mathbb{N}}$ of elements of $\hat{\mathfrak{A}}_c$ s.t. $\{A_m(f)\}_{m \in \mathbb{N}}$ tends to $A(f)$ in norm. From estimate (5.20) we obtain

$$\int d^s p |\varphi(\widetilde{A_m(f)}(\vec{p}))|^2 < \infty \quad (5.26)$$

for any $\varphi \in \hat{\mathfrak{A}}_*$. This implies that $A(f) \in \mathfrak{A}_{ac}$ i.e. $[A(f)] = 0$.

To prove the second part of (c), one shows that for any $\varphi \in \mathfrak{A}_{*,c}$ the corresponding element $[\varphi] \in \mathfrak{A}_{*,sc}$ satisfies $[\varphi_f] = 0$ for any $f \in S(\mathbb{R}^s)$ s.t. $\text{supp } \tilde{f} \cap \{0\} = \emptyset$. The argument is analogous as above. \square

Part (c) of the above theorem states that $\text{Sp}_{sc}\beta$ and $\text{Sp}_{sc}\beta^*$ are either empty or consist only of $\{0\}$. It is an interesting question, whether this latter possibility can be excluded in general. As a step in this direction, we show in the Appendix, that in theories complying with a timelike asymptotic abelianess condition, introduced in [BWa92],

$$\text{Sp}_{sc}\beta = \text{Sp}_{sc}\beta^* = \emptyset \quad (5.27)$$

for $\hat{\mathfrak{A}}$ slightly smaller than \mathfrak{A}_{loc} chosen here. This includes, in particular, the theory of scalar, non-interacting massive and massless particles⁶. In the next section we provide further evidence for triviality of the singular-continuous spectra of β and β^* : We propose a regularity condition, suitable for massive theories, which implies (5.27). We also show that this condition guarantees the existence of particles, if the theory contains a stress-energy tensor.

6 Structure of the continuous spectrum and the particle content in QFT

In the present section, which is based on Section 2.3 of [Dy08.2], we augment the general postulates of quantum field theory, adopted in the previous section, by Conditions $L^{(1)}$ and T stated below. The former is a regularity condition, restricting the structure of the continuous spectrum of α near zero, while the latter encodes the presence of a stress-energy tensor among the pointlike-localized fields of the theory.

⁶In the massless case for $s \geq 3$.

We will show that $\text{Sp}_{\text{sc}}\beta = \text{Sp}_{\text{sc}}\beta^* = \emptyset$ in theories complying with Condition $L^{(1)}$. If, in addition, Condition T is satisfied, we demonstrate that the theory describes particles in the sense of non-zero asymptotic functionals.

In order to formulate Condition $L^{(1)}$, we have to introduce some terminology: We define, for any $E \geq 0$ and $C \in \mathfrak{A}$, the (possibly infinite) quantity

$$\|C\|_{E,1} := \sup_{\omega \in S_E} \int d^s x |\omega(\beta_{\vec{x}}(C))|, \quad (6.28)$$

and introduce the following subspace of \mathfrak{A}

$$\mathfrak{A}^{(1)} := \{ C \in \mathfrak{A} \mid \forall_{E \geq 0} \|C\|_{E,1} < \infty \} \quad (6.29)$$

which is a natural domain for the asymptotic functionals $\sigma_{\omega}^{(+)}$ mentioned in the Introduction and defined precisely in (6.36) below. To study the properties of this subspace, we introduce two useful concepts: First, an operator $B \in \mathfrak{A}$ is called energy-decreasing, if its Arveson spectrum w.r.t. the group of spacetime translation automorphisms does not intersect with the closed forward lightcone i.e. $\text{Sp}^B \alpha \cap \overline{V}_+ = \emptyset$. Second, an observable $B \in \mathfrak{A}$ is called almost local, if there exists a net of local operators $\{ B_r \in \mathfrak{A}(\mathcal{O}(r)) \mid r > 0 \}$, s.t. for any $k \in \mathbb{N}_0$

$$\lim_{r \rightarrow \infty} r^k \|B - B_r\| = 0, \quad (6.30)$$

where $\mathcal{O}(r)$ is a double cone of radius r , centered at the origin. After this preparation we state a result, due to Buchholz [Bu90], which guarantees non-triviality of $\mathfrak{A}^{(1)}$ in any local, relativistic quantum field theory.

Theorem 6.1. [Bu90] *Let $B \in \mathfrak{A}$ be almost local and energy-decreasing. Then, for any $E \geq 0$, there holds $\|B^* B\|_{E,1} < \infty$.*

Our regularity condition specifies another class of observables from $\mathfrak{A}^{(1)}$. These operators are of the form $A(g) = \int dt g(t) \alpha_t(A)$, where $A \in \hat{\mathfrak{A}}_{\text{c}}$ (see definition (5.15)) and \tilde{g} is supported in a small neighborhood of zero. More precisely:

Condition $L^{(1)}$: There exists $\mu > 0$ s.t. for any $g \in S(\mathbb{R})$ with $\text{supp } \tilde{g} \subset] - \mu, \mu[$ and $A \in \hat{\mathfrak{A}}_{\text{c}}$,

$$(a) \quad A(g) \in \mathfrak{A}^{(1)},$$

$$(b) \quad \|A(g)\|_{E,1} \leq c_{l,E,r} \|R^l A R^l\| \|g\|_1, \text{ for all } E \geq 0, l \geq 0,$$

where $R = (1 + H)^{-1}$ and $r > 0$ is s.t. $A \in \mathfrak{A}(\mathcal{O}(r))$.

This condition has been verified in massive scalar free field theory in Appendix D of [Dy08.2], so it is consistent with the basic postulates of quantum field theory. The quantitative part (b) of this criterion is needed in Theorem 6.3 below to prove the existence of non-trivial asymptotic functionals in theories admitting a stress-energy tensor. On the other hand, the qualitative part (a) suffices to conclude that the singular-continuous spectrum of the space translation automorphisms is empty.

Theorem 6.2. *Let $\mathbb{R}^s \ni \vec{x} \rightarrow \beta_{\vec{x}}$ be the group of space translation automorphisms acting on the algebra of observables \mathfrak{A} in a quantum field theory admitting a normal vacuum state ω_0 and satisfying Condition $L^{(1)}(a)$. Let \mathfrak{A}_* , $\hat{\mathfrak{A}}_*$ and $\hat{\mathfrak{A}}$ be given by (5.4). Then $\text{Sp}_{\text{sc}}\beta = \text{Sp}_{\text{sc}}\beta^* = \emptyset$.*

Proof. Suppose that $A \in \hat{\mathfrak{A}}_{\text{c}}$. To show that $A \in \mathfrak{A}_{\text{ac}}$, it suffices to verify that for any $E \geq 0$ and $\omega \in S_E$

$$\int d^s x |\omega(\beta_{\vec{x}}(A))|^2 < \infty. \quad (6.31)$$

(This follows from the Plancherel theorem and the fact that the distributions $\mathbb{R}^s \ni \vec{p} \rightarrow \omega(\tilde{A}(\vec{p}))$ are compactly supported). We fix $E \geq \mu$, where μ appeared in Condition $L^{(1)}$, and choose a function $f \in S(\mathbb{R})$ s.t. $\tilde{f} = (2\pi)^{-\frac{1}{2}}$ on $[-E, E]$ and $\text{supp } \tilde{f} \subset [-2E, 2E]$. With the help of a smooth partition of unity we can decompose f as follows: $f = f_- + f_+ + f_0$, where $\text{supp } \tilde{f}_- \subset [-2E, -\mu/2]$, $\text{supp } \tilde{f}_+ \subset [\mu/2, 2E]$, and $\text{supp } \tilde{f}_0 \subset]-\mu, \mu[$. Then

$$P_E A P_E = P_E A(f) P_E = P_E A(f_-) P_E + P_E A(f_+) P_E + P_E A(f_0) P_E, \quad (6.32)$$

where the first equality is a consequence of relation (4.23). By Condition $L^{(1)}(a)$, $A(f_0)$ satisfies the bound (6.31). To the remaining terms we can apply Theorem 6.1, since both $A(f_-)$ and $A(f_+)^*$ are almost local and energy-decreasing. This latter fact follows from the equality

$$\widetilde{A(f_-)}(p) = (2\pi)^{\frac{1}{2}} \tilde{f}_-(p^0) \tilde{A}(p^0, \vec{p}) \quad (6.33)$$

which implies that the support of $\widetilde{A(f_-)}$ does not intersect with the closed forward lightcone. (An analogous argument applies to $A(f_+)^*$). We obtain for any $\omega \in S_E$

$$\begin{aligned} \int d^s x |\omega(\beta_{\vec{x}}(A(f_-)))|^2 &\leq \sup_{\omega' \in S_E} \int d^s x \omega'(\beta_{\vec{x}}(A(f_-)^* A(f_-))) \\ &= \|A(f_-)^* A(f_-)\|_{E,1}, \end{aligned} \quad (6.34)$$

where the last expression is finite by Theorem 6.1. Since an analogous estimate holds for $A(f_+)$, we conclude that $\hat{\mathfrak{A}}_{\text{c}} \subset \mathfrak{A}_{\text{ac}}$ and therefore $\mathfrak{A}_{\text{c}} = \mathfrak{A}_{\text{ac}}$ i.e. $\text{Sp}_{\text{sc}}\beta = \emptyset$. Now suppose that $\varphi \in \hat{\mathfrak{A}}_{*,\text{c}}$. Then, by (6.31), for any $A \in \hat{\mathfrak{A}}$

$$\int d^s x |\beta_{\vec{x}}^* \varphi(A)|^2 < \infty, \quad (6.35)$$

which implies that $\varphi \in \mathfrak{A}_{*,\text{ac}}$. We conclude that $\text{Sp}_{\text{sc}}\beta^* = \emptyset$. \square

Proceeding to particle aspects of the theory, we note that the space $\mathfrak{A}^{(1)}$, equipped with the family of seminorms $\{\|\cdot\|_{E,1} \mid E \geq 0\}$, is a locally convex Hausdorff space and we call the corresponding topology $T^{(1)}$. (This is established as in Section 2.2 of [Po04.1]). We define, for any $\omega \in S_E$, a net $\{\sigma_{\omega}^{(t)}\}_{t \in \mathbb{R}_+}$ of functionals on $\mathfrak{A}^{(1)}$ given by

$$\sigma_{\omega}^{(t)}(C) := \int d^s x \omega(\alpha_{(t,\vec{x})}(C)), \quad C \in \mathfrak{A}^{(1)}. \quad (6.36)$$

This net satisfies the uniform bound $|\sigma_\omega^{(t)}(C)| \leq \|C\|_{E,1}$. Therefore, by the Alaoglu-Bourbaki theorem (see [Ja], Section 8.5), it has weak limit points $\sigma_\omega^{(+)}$ in the topological dual of $(\mathfrak{A}^{(1)}, T^{(1)})$ which we call the asymptotic functionals. The set of such functionals

$$\mathfrak{P} := \{ \sigma_\omega^{(+)} \mid \omega \in S_E \text{ for some } E \geq 0 \} \quad (6.37)$$

will be called the particle content of the theory. This terminology was justified in the Introduction, where we argued that the asymptotic functionals should carry information about all the (infra-)particle types appearing in the theory. A general argument for the existence of non-zero asymptotic functionals has been given to date only for theories of Wigner particles [AH67]. It is now our goal to show that $\mathfrak{P} \neq \{0\}$ not relying on the Wigner concept of a particle.

Since our argument is based on the existence of a stress-energy tensor, which is postulated in Condition T below, we recall the definition and simple properties of pointlike-localized fields: We set $R = (1 + H)^{-1}$ and introduce the space of normal functionals with polynomially damped energy

$$B(\mathcal{H})_{*,\infty} := \bigcap_{l \geq 0} R^l B(\mathcal{H})_* R^l. \quad (6.38)$$

We equip this space with the locally convex topology given by the norms $\|\cdot\|_l = \|R^{-l} \cdot R^{-l}\|$ for $l \geq 0$. The field content of the theory is defined as follows [FH81]

$$\Phi_{\text{FH}} := \{ \phi \in (B(\mathcal{H})_{*,\infty})^* \mid R^l \phi R^l \in \bigcap_{r>0} \{ R^l \mathfrak{A}(\mathcal{O}(r)) R^l \}^{\text{w-cl}} \text{ for some } l \geq 0 \}, \quad (6.39)$$

where w-cl denotes the weak closure in $B(\mathcal{H})$. Since the normal vacuum state ω_0 is an element of $B(\mathcal{H})_{*,\infty}$, we can define

$$\Phi_{\text{FH},c} := \{ \phi \in \Phi_{\text{FH}} \mid \omega_0(\phi) = 0 \}. \quad (6.40)$$

There holds the following useful approximation property for the pointlike-localized fields which is due to Bostelmann [Bos05]: For any $\phi \in \Phi_{\text{FH},c}$ there exists $l \geq 0$ and a net $A_r \in \mathfrak{A}(\mathcal{O}(r))$, $r > 0$, $\omega_0(A_r) = 0$, s.t.

$$\lim_{r \rightarrow 0} \|R^l(A_r - \phi)R^l\| = 0. \quad (6.41)$$

Making use of Condition $L^{(1)}(\text{b})$, we also obtain, for any time-smearing function $g \in S(\mathbb{R})$ s.t. $\text{supp } \tilde{g} \subset]-\mu, \mu[$,

$$\lim_{r \rightarrow 0} \|A_r(g) - \phi(g)\|_{E,1} = 0. \quad (6.42)$$

This implies, in particular, that $\|\phi(g)\|_{E,1} < \infty$ for any $\phi \in \Phi_{\text{FH},c}$, which prepares the ground for our next assumption:

Condition T : There exists a field $T^{00} \in \Phi_{\text{FH},c}$ which satisfies

$$\int d^s x \omega(\beta_{\vec{x}}(T^{00}(g))) = \omega(H), \quad \omega \in S_E, \quad (6.43)$$

for any $E \geq 0$ and any time-smearing function $g \in S(\mathbb{R})$ s.t. $\tilde{g}(0) = (2\pi)^{-\frac{1}{2}}$ and $\text{supp } \tilde{g} \subset]-\mu, \mu[$, where μ appeared in Condition $L^{(1)}$.

This condition holds, in particular, in massive scalar free field theory as shown in Section B.2 of [Dy08.2]. With Conditions $L^{(1)}$ and T at hand, we are ready to prove the existence of non-zero asymptotic functionals.

Theorem 6.3. *Suppose that a quantum field theory, admitting a normal vacuum state ω_0 , satisfies Conditions $L^{(1)}$ and T and let $\omega \in S_E$ be s.t. $\omega(H) > 0$. Then all the limit points $\sigma_\omega^{(+)}$ are non-zero.*

Proof. We choose $g \in S(\mathbb{R})$ as in Condition T and $0 < \varepsilon \leq \frac{1}{2}|\omega(H)|$. Making use of Condition $L^{(1)}$ (b) and relation (6.42), we can find $C \in \mathfrak{A}^{(1)}$ s.t. $\|T^{00}(g) - C\|_{E,1} \leq \varepsilon$. Then, exploiting Condition T and invariance of H under time translations, we obtain

$$|\omega(H)| = \left| \int d^s x \omega(\alpha_{(t,\vec{x})}(T^{00}(g))) \right| \leq \varepsilon + \left| \int d^s x \omega(\alpha_{(t,\vec{x})}(C)) \right|. \quad (6.44)$$

Thus we arrive at a positive lower bound $\omega(H) \leq 2|\sigma_\omega^{(t)}(C)|$ which is uniform in t . \square

We emphasize that we have proven more than non-triviality of the particle content - we have verified that every physical state, with non-zero mean energy, gives rise to a non-trivial asymptotic functional. On the other hand, we did not touch upon the problem of convergence of the nets $\{\sigma_\omega^{(t)}\}_{t \in \mathbb{R}_+}$ which is essential for their physical interpretation in terms of particle measurements. The question, if the energy of the state ω can be reconstructed from the four-momenta characterizing the pure particle weights, appearing in the decomposition (1.3) of $\sigma_\omega^{(+)}$, is another important open problem. Such a result would be an essential step towards a model-independent understanding of the problem of asymptotic completeness in quantum field theory (cf. discussion in [Bu94]). Regularity properties of the continuous spectrum of α should be of relevance to the study of these issues.

7 Conclusions and outlook

In this paper we defined and analyzed the continuous Arveson spectrum of a group of isometries α acting on a Banach space \mathfrak{A} . We introduced new notions of the absolutely continuous and singular continuous spectra of α and defined the corresponding spectral spaces. By studying relations between the spectral concepts on the side of α and α^* we found necessary and sufficient conditions for the pure-point and continuous subspaces to span the entire Banach space. The sufficient conditions have a natural formulation, if \mathfrak{A} is a unital C^* -algebra equipped with a distinguished, invariant state ω_0 . In this setting we established relations between the continuous spectrum of α and the spectrum of the implementing group of unitaries in the GNS representation induced by ω_0 . We verified that in any quantum field theory, admitting a normal vacuum state, the group of spacetime translation automorphisms fits into this algebraic framework. We concluded that in a theory of Wigner particles the singular-continuous spectrum of α contains the corresponding mass hyperboloid, while the multiparticle spectrum of the energy-momentum

operators is included in the absolutely continuous spectrum of α^* . Moreover, we found conditions on the continuous spectrum of α in a neighborhood of zero which, on the one hand, imply triviality of the singular-continuous spectrum of the space translation automorphisms, on the other hand entail the existence of particles, if the theory contains a stress-energy tensor.

While this latter assumption is physically reasonable, we feel that the presence of (infra-)particles, which is a large-scale phenomenon, should not depend on short-distance properties, like the existence of certain pointlike-localized fields. It should be possible to find general necessary and sufficient conditions for non-triviality of the particle content in terms of some spectral properties of the group of translation automorphisms. In the second step of the analysis these criteria should be related to physical properties of the theory (e.g. the phase space structure or the existence of constants of motion). Since pure particle weights corresponding to different momenta of an inraparticle can give rise to inequivalent representations of the algebra of observables, it may be necessary to look for more general spectral concepts than these introduced in the present work. Such notions should not depend on the choice of a specific vacuum state, but rather carry information about some large class of positive energy representations. First steps in this direction are taken in the Appendix, where we choose as \mathfrak{A}_* the space of energetically accessible functionals (see definition (A.5)), rather than the predual in some vacuum representation. We hope to return to these problems in a future publication.

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Appendix: Spectral theory of automorphism groups in QFT in the absence of normal vacuum states

Theorem 3.3 provides sufficient conditions for the following equalities

$$\mathfrak{A} = \mathfrak{A}_{\text{pp}} \oplus \mathfrak{A}_{\text{c}}, \quad (\text{A.1})$$

$$\mathfrak{A}_* = \mathfrak{A}_{*,\text{pp}} \oplus \mathfrak{A}_{*,\text{c}} \quad (\text{A.2})$$

which we verified in Section 5 in a large class of examples. In the first part of this Appendix we describe situations where decompositions (A.1), (A.2) fail. In particular, we show that in the case of the group of space translation automorphisms β , acting on the algebra of observables in quantum field theory, equality (A.1) fails, if \mathfrak{A}_* contains more than one vacuum state. This occurs in some low-dimensional massless theories, if \mathfrak{A}_* is chosen as the space of energetically accessible functionals, defined in (A.5) below. On the other hand, for theories complying with Condition A, stated below, equalities (A.1), and (A.2) hold for this choice of \mathfrak{A}_* . In the second part of the Appendix, which relies on results from [BWa92], we analyze briefly the continuous spectra of β and β^* . It turns out that their singular-continuous parts are empty in this setting.

We start from the following simple observation:

Proposition A.1. *Let $\mathbb{R}^s \ni \vec{x} \rightarrow \beta_{\vec{x}}$ be the group of space translation automorphisms acting on the algebra of observables \mathfrak{A} in a quantum field theory (possibly without normal vacuum states). Then $\mathfrak{A}_{\text{pp}} = \text{Span}\{I\}$ for any \mathfrak{A}_* satisfying Condition S.*

Proof. Suppose that $A \in \mathfrak{A}$ is an eigenvector i.e.

$$\beta_{\vec{x}}(A) = e^{-i\vec{q}\vec{x}} A, \quad \vec{x} \in \mathbb{R}^s \quad (\text{A.3})$$

for some $\vec{q} \in \mathbb{R}^s$. Then A belongs to the center of \mathfrak{A} , since locality gives

$$\|[A, B]\| = \lim_{|\vec{x}| \rightarrow \infty} \|[\alpha_{\vec{x}}(A), B]\| = 0, \quad B \in \mathfrak{A}. \quad (\text{A.4})$$

The irreducibility assumption ensures that the center of \mathfrak{A} consists only of multiples of the identity. \square

In view of Theorem 3.2 (a), equality (A.2) fails, in particular, if $\dim \tilde{\mathfrak{A}}_*(\{0\}) < \dim \tilde{\mathfrak{A}}(\{0\})$. Let us consider the group of space translation automorphisms β acting on the algebra of observables \mathfrak{A} in a quantum field theoretic model which does not admit normal vacuum states, (see [BHS63] for an example). Thus, choosing $\mathfrak{A}_* = B(\mathcal{H})_*$, we obtain $\dim \tilde{\mathfrak{A}}_*(\{0\}) = 0$, whereas Proposition A.1 gives $\dim \tilde{\mathfrak{A}}(\{0\}) = 1$.

On the other hand, equality (A.1) fails when $\dim \tilde{\mathfrak{A}}_*(\{0\}) > \dim \tilde{\mathfrak{A}}(\{0\})$. To exhibit an example, let us choose as \mathfrak{A}_* the space of energetically accessible functionals

$$B(\mathcal{H})_*^{(\text{a})} := \text{Span}\left\{ \bigcup_{E \geq 0} S_E^{\text{w}^*-\text{cl}} \right\}^{\text{n-cl}}, \quad (\text{A.5})$$

where w^* -cl denotes the closure in the weak* topology of \mathfrak{A}^* . (Clearly, this space satisfies Condition S). It is well known, that massless free field theory in $s = 2$ dimensional space has an infinite family of vacuum states in $B(\mathcal{H})_*^{(a)}$ [BWa92]. Hence, by Theorem A.1, $\mathfrak{A} \neq \mathfrak{A}_{\text{pp}} \oplus \mathfrak{A}_c$ in this situation.

However, there exists a large class of theories, in which the choice $\mathfrak{A}_* = B(\mathcal{H})_*^{(a)}$ entails equalities (A.1) and (A.2). These are, in particular, models which satisfy the following asymptotic abelianess assumption, proposed by Buchholz and Wenzel [BWa92].

Condition A: There exists a norm dense subspace $\mathcal{D} \subset \mathfrak{A}_{\text{loc}}$ s.t. for any $A \in \mathcal{D}$ there exists some positive number $1 \leq r < s$, s.t. for all $\Phi \in \mathcal{H}$

$$\sup_{x_0} \int d^s x \|[A^*, \alpha_{x_0, \vec{x}}(A)]\Phi\|^r < \infty. \quad (\text{A.6})$$

These authors have shown that Condition A holds in massive (for $s \geq 1$) and massless (for $s \geq 3$) free field theory. Moreover, it was verified in [BWa92] that in theories complying with Condition A there exists a distinguished vacuum state ω_0 in $B(\mathcal{H})_*^{(a)}$ s.t. for $A \in \mathfrak{A}$

$$w^* - \lim_{|\vec{x}| \rightarrow \infty} \beta_{\vec{x}}(A) = \omega_0(A)I. \quad (\text{A.7})$$

(Other conditions which imply this property can be found in [Dy08.1, Dy09]). By a slight modification of the discussion from Section 4 of [BWa92], we obtain that for any $A \in \ker \omega_0$, $\varphi \in B(\mathcal{H})_*^{(a)}$ and $\vec{q} \in \mathbb{R}^s$

$$\lim_{K \nearrow \mathbb{R}^s} \frac{1}{|K|} \int_K e^{i\vec{q}\vec{x}} \varphi(\beta_{\vec{x}}(A)) d^s x = 0. \quad (\text{A.8})$$

Hence $\ker \omega_0 \subset \mathfrak{A}_c$ and, consequently, ω_0 is the unique element of $B(\mathcal{H})_*^{(a)}$ invariant under the action of β^* . Thus the decomposition of \mathfrak{A} and \mathfrak{A}_* into the pure-point and continuous subspaces is given by Theorem 3.3. We summarize:

Theorem A.2. *Let $\mathbb{R}^s \ni \vec{x} \rightarrow \beta_{\vec{x}}$ be the group of space translation automorphisms acting on the algebra of observables \mathfrak{A} in a quantum field theory satisfying Condition A. Let $\mathfrak{A}_* = B(\mathcal{H})_*^{(a)}$ and let ω_0 be the unique vacuum state in $B(\mathcal{H})_*^{(a)}$. Then $\ker \omega_0 \subset \mathfrak{A}_c$ and, consequently,*

$$(a) \quad \mathfrak{A} = \mathfrak{A}_{\text{pp}} \oplus \mathfrak{A}_c, \text{ where } \mathfrak{A}_{\text{pp}} = \text{Span}\{I\}, \mathfrak{A}_c = \ker \omega_0,$$

$$(b) \quad \mathfrak{A}_* = \mathfrak{A}_{*,\text{pp}} \oplus \mathfrak{A}_{*,c}, \text{ where } \mathfrak{A}_{*,\text{pp}} = \text{Span}\{\omega_0\}, \mathfrak{A}_{*,c} = \ker I.$$

The analysis of the absolutely continuous and singular-continuous spectrum is performed similarly as in Theorem 5.4. However, the norm dense subspaces $\hat{\mathfrak{A}} \subset \mathfrak{A}$ and $\hat{\mathfrak{A}}_* \subset \mathfrak{A}_*$, are now chosen as follows

$$\hat{\mathfrak{A}} = \mathcal{D}, \quad (\text{A.9})$$

$$\hat{\mathfrak{A}}_* = B(\mathcal{H})_{*,\text{bd}}^{(a)} := \text{Span}\left\{\bigcup_{E \geq 0} S_E^{w^*\text{-cl}}\right\}, \quad (\text{A.10})$$

where \mathcal{D} appeared in Condition A. In the present case we are able to show that the singular-continuous spectra of β and β^* are empty.

Theorem A.3. *Let $\mathbb{R}^s \ni \vec{x} \rightarrow \beta_{\vec{x}}$ be the group of space translation automorphisms acting on the algebra of observables \mathfrak{A} in a quantum field theory satisfying Condition A. Let \mathfrak{A}_* , $\hat{\mathfrak{A}}$ and $\hat{\mathfrak{A}}_*$ be given by (A.5), (A.9) and (A.10), respectively. Then there holds:*

$$(a) \text{ Sp}_{\text{pp}}\beta = \text{Sp}_{\text{pp}}\beta^* = \{0\},$$

$$(b) \text{ Sp}_{\text{ac}}\beta = \text{Sp}_{\text{ac}}\beta^* = \mathbb{R}^s,$$

$$(c) \text{ Sp}_{\text{sc}}\beta = \text{Sp}_{\text{sc}}\beta^* = \emptyset.$$

Proof. Part (a) follows from Theorem A.2. To prove statements (b) and (c), we define the subspaces

$$\hat{\mathfrak{A}}_c^{(a)} := \{ A \in \hat{\mathfrak{A}} \mid \omega_0(A) = 0 \}, \quad (\text{A.11})$$

$$\hat{\mathfrak{A}}_{*,c}^{(a)} := \{ \varphi \in \hat{\mathfrak{A}}_* \mid \varphi(I) = 0 \} \quad (\text{A.12})$$

which are norm dense in \mathfrak{A}_c and $\mathfrak{A}_{*,c}$, respectively. We choose $A \in \hat{\mathfrak{A}}_c^{(a)}$, $f \in S(\mathbb{R}^s)$ s.t. \tilde{f} vanishes in some neighborhood of zero and set $A(f) := \int d^s x f(\vec{x}) \beta_{\vec{x}}(A)$. Then we obtain from Lemma 2.1 of [BWa92] the bound

$$\|P_E A(f) P_E\| \leq c \left(\int d^s p |\vec{p}|^{-(s-\varepsilon)} |\tilde{f}(\vec{p})|^2 \right)^{1/2} \quad (\text{A.13})$$

for some constants $c > 0$, $\varepsilon > 0$ independent of f . Noting that for any $\omega \in S_E^{\text{w*}-\text{cl}}$, $|\omega(A(f))| \leq \|P_E A(f) P_E\|$ holds, making use of the assumption that β acts strongly continuously on \mathfrak{A} to exchange the action of the state ω with integration and proceeding as in [BWa92], p.581, we obtain

$$\omega(\beta_{\vec{x}}(A)) = l(\vec{x}) + \omega'_0(A), \quad (\text{A.14})$$

where $\tilde{l} \in L^1(\mathbb{R}^s, d^s p)$ and $\omega'_0 = \text{w*}-\lim_{|\vec{x}| \rightarrow \infty} \beta_{\vec{x}}^* \omega$ is an element of $S_E^{\text{w*}-\text{cl}}$, invariant under the action of β^* . So, by Theorem A.2 (b), $\omega'_0 = \omega_0$ and consequently

$$\forall_{\substack{A \in \hat{\mathfrak{A}}_c^{(a)} \\ \varphi \in \hat{\mathfrak{A}}_{*,c}^{(a)}}} \int d^s p |\varphi(\tilde{A}(\vec{p}))| < \infty. \quad (\text{A.15})$$

We conclude, that $\mathfrak{A}_c = \mathfrak{A}_{\text{ac}}$ and $\mathfrak{A}_{*,c} = \mathfrak{A}_{*,\text{ac}}$, which proves (c). Part (b) follows from the inclusions

$$\overline{\bigcup_{\substack{A \in \hat{\mathfrak{A}}_c^{(a)} \\ \varphi \in \hat{\mathfrak{A}}_*}} \text{supp } \varphi(\tilde{A}(\cdot))} \subset \text{Sp}_{\text{ac}}\beta, \quad (\text{A.16})$$

$$\overline{\bigcup_{\substack{A \in \mathfrak{A} \\ \varphi \in \hat{\mathfrak{A}}_{*,c}^{(a)}}} \text{supp } \varphi(\tilde{A}(\cdot))} \subset \text{Sp}_{\text{ac}}\beta^* \quad (\text{A.17})$$

and from Lemma 5.3. To apply this latter fact, we note that $\hat{\mathfrak{A}}_c^{(a)} \subset \hat{\mathfrak{A}}_c$ and $\hat{\mathfrak{A}}_{*,c}^{(a)} \supset \hat{\mathfrak{A}}_{*,c}$, where $\hat{\mathfrak{A}}_c$ and $\hat{\mathfrak{A}}_{*,c}$ are given by (5.15) and (5.16), respectively. \square

The above result has the following immediate corollary: After adding Condition A to the assumptions of Theorem 5.4 and choosing $\hat{\mathfrak{A}} = \mathcal{D}$, part (c) of this theorem can be strengthened to $\text{Sp}_{\text{sc}}\beta = \text{Sp}_{\text{sc}}\beta^* = \emptyset$.

References

- [Ar] H. Araki, *Mathematical Theory of Quantum Fields*. Oxford University Press 1999.
- [AH67] H. Araki and R. Haag, *Collision cross sections in terms of local observables*. Commun. Math. Phys. **4** (1967), 77-91.
- [AB97] W. Arendt and C. J. K. Batty, *Almost periodic solutions of first and second order Cauchy problems*. J. Differential Equations **137** (1997), 363-383.
- [Ar74] W. Arveson, *On groups of automorphisms of operator algebras*. J. Funct. Anal. **15** (1974), 217-243.
- [Ar82] W. Arveson, *The harmonic analysis of automorphism groups*. In Operator algebras and applications, Part I (Kingston, Ont., 1980), Proc. Sympos. Pure Math., 38, Amer. Math. Soc., Providence, R.I., 1982.D., pp. 199-269.
- [Ba78] A. G. Baskakov, *Spectral criteria for almost periodicity of solutions of functional equations*. Math. Notes **24** (1978), 606-612.
- [BHS63] H. J. Borchers, R. Haag and B. Schroer, *The vacuum state in quantum field theory*. Nuovo Cimento **29** (1963), 148-162.
- [Bos05] H. Bostelmann, *Phase space properties and the short distance structure in quantum field theory*. J. Math. Phys. **46** (2005), 052301-052318.
- [Bu82] D. Buchholz, *The physical state space of quantum electrodynamics*. Commun. Math. Phys. **85** (1982), 49-71.
- [Bu86] D. Buchholz, *Gauss' law and the infraparticle problem*. Phys. Lett. B **174** (1986), 331-334.
- [Bu90] D. Buchholz, *Harmonic analysis of local operators*. Commun. Math. Phys. **129** (1990), 631-641.
- [Bu94] D. Buchholz, *On the manifestations of particles*. pp.177-202 in: Mathematical Physics Towards the 21st Century. Proceedings Beer-Sheva 1993, R.N. Sen and A. Gersten Eds., Ben-Gurion University of the Negev Press 1994.
- [BF82] D. Buchholz and K. Fredenhagen, *Locality and the structure of particle states*. Commun. Math. Phys. **84** (1982), 1-54.

- [BPS91] D. Buchholz, M. Porrmann and U. Stein, *Dirac versus Wigner: Towards a universal particle concept in quantum field theory*. Phys. Lett. B **267** (1991), 377-381.
- [BWa92] D. Buchholz and R. Wanzenberg, *The realm of the vacuum*. Commun. Math. Phys. **143** (1992), 577-589.
- [CFP07] T. Chen, J. Fröhlich and A. Pizzo, *Infraparticle scattering states in non-relativistic QED: I. The Bloch-Nordsieck paradigm*. Preprint: arXiv:0709.2493.
- [De93] J. Dereziński, *Asymptotic completeness of long-range N -body quantum systems*. Ann. of Math. **138** (1993), 427-476.
- [DG00] J. Dereziński and C. Gerard, *Spectral and scattering theory of spatially cut-off $P(\phi)_2$ Hamiltonians*. Commun. Math. Phys. **213** (2000), 39-125.
- [Dy05] W. Dybalski, *Haag-Ruelle scattering theory in presence of massless particles*. Lett. Math. Phys. **72** (2005), 27-38.
- [Dy08.1] W. Dybalski, *A sharpened nuclearity condition and the uniqueness of the vacuum in QFT*. Commun. Math. Phys. **283** (2008), 523-542.
- [Dy08.2] W. Dybalski, *Spectral theory of automorphism groups and particle structures in quantum field theory*. PhD thesis, Universität Göttingen (2008). Preprint: <http://webdoc.sub.gwdg.de/diss/2009/dybalski/>
- [Dy09] W. Dybalski, *Coincidence arrangements of local observables and uniqueness of the vacuum in QFT*. J. Phys. A **42** (2009), 365201-365223.
- [En78] V. Enss, *Asymptotic completeness for quantum mechanical potential scattering*. Commun. Math. Phys. **61** (1978), 285-291.
- [Ev76] D. Evans, *On the spectrum of a one parameter strongly continuous representation*. Math. Scand. **39** (1976), 80-82.
- [Ex99] R. Exel, *Unconditional integrability for dual actions*. Bol. Soc. Brasil. Mat. (N.S.) **30** (1999), 99-124.
- [Ex00] R. Exel, *Morita-Rieffel equivalence and spectral theory for integrable automorphism groups of C^* -algebras*. J. Funct. Anal. **172** (2000), 404-465.
- [FMS79] J. Fröhlich, G. Morchio and F. Strocchi, *Infrared problem and spontaneous breaking of the Lorentz group in QED*. Phys. Lett. B **89** (1979), 61-64.
- [FGS04] J. Fröhlich, M. Griesemer and B. Schlein, *Asymptotic completeness for Compton scattering*. Commun. Math. Phys. **252** (2004), 415-476.
- [FH81] K. Fredenhagen and J. Hertel, *Local algebras of observables and pointlike localized fields*. Commun. Math. Phys. **80** (1981), 555-561.

- [Gr90] G. M. Graf, *Asymptotic completeness for N -body short-range quantum systems: a new proof*. Commun. Math. Phys. **132** (1990), 73-101.
- [Ha] R. Haag, *Local Quantum Physics*. Springer 1996.
- [He07] A. Herdegen, *Infrared problem and spatially local observables in electrodynamics*. Ann. Henri Poincaré **9** (2008), 373-401.
- [Hu99] Sen-Zhong Huang, *Completeness of eigenvectors of group representations of operators whose Arveson spectrum is scattered*. Proc. Amer. Math. Soc. **127** (1999), 1473-1482.
- [Ja] H. Jarchow, *Locally Convex Spaces*. B. G. Teubner, 1981.
- [Joh91] K. Johannsen, *Teilchenaspekte im Schroermodell*. Diplomarbeit, Universität Hamburg, 1991.
- [Jo92] P. E. T. Jorgensen, *Spectral theory of one-parameter groups of isometries*. J. Math. Anal. Appl. **168** (1992), 131-146.
- [Jo82] P. E. T. Jorgensen, *Spectral theory for infinitesimal generators of one-parameter groups of isometries: The mini-max principle and compact perturbations*. J. Math. Anal. Appl. **90** (1982), 347-370.
- [Le08] G. Lechner, *Construction of quantum field theories with factorizing S -matrices*. Commun. Math. Phys. **277** (2008), 821-860.
- [Lo77] R. Longo, *Some aspects of C^* -dynamics*. In: "Algèbres d'opérateurs et leur applications en physique mathématique" Colloques Internationaux du C.N.R.S. n 274, 1977.
- [Me01] R. Meyer, *Generalized fixed point algebras and square-integrable group actions*. J. Funct. Anal. **186** (2001), 167-195.
- [Pi05] A. Pizzo, *Scattering of an infraparticle: the one particle sector in Nelson's massless models*. Ann. Henri Poincaré **6** (2005), 553-606.
- [Pe] G. K. Pedersen, *C^* -Algebras and Their Automorphism Groups*. Academic Press, 1979.
- [Po04.1] M. Porrmann, *Particle weights and their disintegration I*. Commun. Math. Phys. **248** (2004), 269-304.
- [Po04.2] M. Porrmann, *Particle weights and their disintegration II*. Commun. Math. Phys. **248** (2004), 305-333.
- [RS1] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. Part I: Functional Analysis*. Academic Press, 1972.
- [Re09] K. Rejzner, *Asymptotic algebra of fields in quantum electrodynamics*. Master's thesis, University of Cracow, 2009.

- [Ri90] M. A. Rieffel, *Proper actions of groups on C^* -algebras*. In: Mappings of operator algebras (Philadelphia, PA, 1988), pp.141-182, Birkhäuser Boston, Boston, MA, 1990.
- [Ru] W. Rudin, *Functional Analysis*. McGraw-Hill, 1977.
- [Sch63] B. Schroer, *Infrateilchen in der Quantenfeldtheorie*. Fortschr. Phys. **11** (1963), 1-32.
- [SiSo87] I. M. Sigal and A. Soffer, *The N -particle scattering problem: asymptotic completeness for short-range systems*. Ann. of Math. **126** (1987), 35-108.
- [Sp] H. Spohn, *Dynamics of Charged Particles and Their Radiation Field*. Cambridge University Press, 2004.
- [Wi39] E. P. Wigner, *On unitary representations of the inhomogeneous Lorentz group*. Ann. Math. **40** (1939), 149-204.